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Equivariant eta forms and equivariant differential K-theory

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Abstract In this paper, for a compact Lie group action, we prove the anomaly formula and the functoriality of the equivariant Bismut-Cheeger eta forms with perturbation operators when the equivariant family index vanishes. In order to prove them, we extend the Melrose-Piazza spectral section and its main properties to the equivariant case and introduce the equivariant version of the Dai-Zhang higher spectral flow for arbitrary dimensional fibers. Using these results, we construct a new analytic model of the equivariant differential K-theory for compact manifolds when the group action has finite stabilizers only, which modifies the Bunck-Schick model of the differential K-theory. This model could also be regarded as an analytic model of the differential K-theory for compact orbifolds. Especially, we answer a question proposed by Bunke and Schick about the well-definedness of the push-forward map.

Keywords equivariant eta form, equivariant differential K-theory, equivariant spectral section, equivariant higher spectral flow, orbifold

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1 Introduction

The differential K-theory is the differential extension of the topological K-theory, whose basic idea is to combine the topological K-theory with the differential form information. It is partly motivated by the study of D-branes in theoretical physics (see e.g., [25,52]). Various models of differential K-theory have been given: Hopkins-Singer [31], Bunke-Schick [16], Freed-Lott [27], Simons-Sullivan [49], Tradler-Wilson-Zeinalian [51], Gorokhovsky-Lott [29], etc. For a detailed survey, see [17].

Until now, the equivariant version of the differential K-theory is not well understood yet. When the group is finite, the equivariant differential K-theory was studied by Szabo-Valentino [50] and Ortiz [46]. In [18], Bunke and Schick extended

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their model to the orbifold case using the language of stacks, which could be regarded as a model of the equivariant differential K-theory when the action has finite stabilizers only.

Inspired by the model of Bunke-Schick [18], as a parallel version, in this paper, we will construct a purely analytic model of the equivariant differential K-theory for compact manifolds when the action has finite stabilizers only, using the local index technique developed in [12]. Moreover, a detailed proof of the well-definedness of the push-forward map is given here, which is a question proposed in [16, 18] and is the main motivation for this new construction. This model is a direct generalization of [16] without using the language of stacks and could also be regarded as an analytic model of the differential K-theory for compact orbifolds.

The study of the differential K-theory is always related to the Bismut-Cheeger eta form [9], which is defined for a family of Dirac operators and is the family extension of the famous Atiyah-Patodi-Singer eta invariant. Usually, the well-definedness of the eta form needs one of the following additional conditions:

1. the kernels of the family of Dirac operators form a vector bundle over the base manifold [9, 19];
2. the family index of the family of Dirac operators vanishes as an element of the K-group of the base manifold [14, 43, 44].

In the model of Freed-Lott [27], the eta form with the first condition is used to define the push-forward map. In the model of Bunke-Schick [16], the eta form under the second condition, which is defined by Bunke [14] using the taming, is used to define the differential K-group. From this point of view, in order to extend the differential K-theory to the equivariant case, we firstly need to extend the Bismut-Cheeger eta form to the equivariant case.

In [34], the author systematically studied the equivariant eta form under the first condition and prove the anomaly formula and the functoriality of them, which should be used to establish an equivariant version of the Freed-Lott model. In this paper, we will establish the properties of the eta form under the second condition, extend them to the equivariant case and use them to construct our model. In order to do finer spectral analysis, we use the notion of the spectral section developed by Melrose and Piazza in [43, 44] and the Dai-Zhang higher spectral flow [20] instead of the taming and the Kasporov KK-theory in [14, 16].

In [43, 44], in order to prove the family index theorem for manifolds with boundary, Melrose and Piazza defined the spectral section and the eta form under the second condition. In [20], using the spectral section, Dai and Zhang introduced the higher spectral flow for a family of Dirac type operators on a family of odd dimensional manifolds. In this paper, we will extend the spectral section, the higher spectral flow and the eta form to the equivariant case. Especially, we will define the higher spectral flow for a family of even dimensional manifolds. Furthermore, we will prove the anomaly formula and the functoriality of equivariant eta forms using the language of equivariant higher spectral flow, which is an analogue of the results in [7, 15, 34] and using the techniques in [19, 39–41]. Note that our proof of the anomaly formula of the eta forms for a family of even dimensional manifolds relies on the funtoriality of equivariant eta forms Theorem 3.18, which is highly nontrivial and is the main technical difficulty of this paper. Since the second condition is a topological condition, there is no additional rigidity assumption in the formulas here.

Let $\pi : W \to B$ be a proper smooth submersion of compact manifolds with orientable fibers Z. Let $TZ = \ker(d\pi)$ be the relative tangent bundle to the fibers

Z with Riemannian metric g^{TZ} and T^HW be a horizontal subbundle of TW, such that $TW = T^HW \oplus TZ$. Let o be an orientation of TZ. Let ∇^{TZ} be the Euclidean connection on TZ defined in (2.14). We assume that TZ has a Spin^c structure. Let L_Z be the complex line bundle associated with the Spin^c structure of TZ with a Hermitian metric h^{L_Z} and a Hermitian connection ∇^{L_Z}. Let (E, h^E) be a $\mathbb{Z}_2$-graded Hermitian vector bundle with a Hermitian connection ∇^E. Let G be a compact Lie group which acts on W and B such that $\pi \circ g = g \circ \pi$ for any $g \in G$. We assume that the G-action preserves everything. The family of G-equivariant geometric data $\mathcal{F} = (W, L_Z, E, o, T^HW, g^{TZ}, h^{L_Z}, \nabla^{L_Z}, h^E, \nabla^E)$ is enough to define the equivariant Bismut superconnection. We call $\mathcal{F}$ an equivariant geometric family over B for short. Let $D(\mathcal{F})$ be the fiberwise Dirac operators of $\mathcal{F}$ defined in (2.23). Let $K_G^i(B)$, $i = 0, 1$, be the equivariant K-group of B. Then the family index map $\mathrm{Ind}(D(\mathcal{F})) \in K_G^*(B)$, where $* = 0$ or 1 corresponds to the even or odd dimensions of fibers Z.

Let $\mathrm{F}_G^0(B)$ (resp. $\mathrm{F}_G^1(B)$) be the set of equivalence classes of isomorphic equivariant geometric families such that the dimensions of all fibers are even (resp. odd). We denote by $\mathcal{F} \sim \mathcal{F}'$ if $\mathrm{Ind}(D(\mathcal{F})) = \mathrm{Ind}(D(\mathcal{F}'))$. The following proposition is proved in [18].

Proposition 1.1. *There is a ring isomorphism*

$$\mathrm{F}_G^*(B)/\sim \,\simeq K_G^*(B). \tag{1.1}$$

Let D be a family of first-order pseudodifferential operators on the fibers of $\mathcal{F}$, which is self-adjoint, fiberwise elliptic and commutes with the G-action. Furthermore, if $\mathcal{F} \in \mathrm{F}_G^1(B)$, we assume that D preserves the $\mathbb{Z}_2$-grading of E; if $\mathcal{F} \in \mathrm{F}_G^0(B)$, we assume that D anti-commutes with the $\mathbb{Z}_2$-grading of $\mathcal{S}(TZ, L_Z)\widehat{\otimes}E$, where $\mathcal{S}(TZ, L_Z)$ is the spinor with respect to the Spin^c structure of TZ. As in [20], we call such D an equivariant B-family on $\mathcal{F}$ (see Definition 3.1). If $\mathrm{Ind}(D) = 0 \in K_G^*(B)$ and at least one component of the fiber has nonzero dimension, there exists an equivariant spectral section P (see Definition 3.2) and a family of smoothing operators A_P associated with P, such that $D + A_P$ is an invertible equivariant B-family (see Proposition 3.3). Let P, Q be equivariant spectral sections, we could define the difference $[P - Q] \in K_G^*(B)$ (see (3.13) and (3.18)).

Let $\mathcal{F}, \mathcal{F}' \in \mathrm{F}_G^1(B)$ (resp. $\mathrm{F}_G^0(B)$) which have the same topological structure, that is, the only differences between them are horizontal subbundles, metrics and connections. Let D_0, D_1 be two equivariant B-families on $\mathcal{F}$, $\mathcal{F}'$ respectively. Let Q_0, Q_1 be equivariant spectral sections of D_0, D_1 respectively. We define the equivariant higher spectral flow $\mathrm{sf}_G\{(D_0, Q_0), (D_1, Q_1)\}$ between the pairs (D_0, Q_0) and (D_1, Q_1) to be an element in $K_G^0(B)$ (resp. $K_G^1(B)$) in Definitions 3.7 and 3.8. Note that when $\mathcal{F}$ is odd, it is the direct extension of the Dai-Zhang higher spectral flow in [20]; when $\mathcal{F}$ is even, it is defined by adding an additional dimension.

Moreover, besides the equivariant geometric family, we could also represent the elements of equivariant K-group as equivariant higher spectral flows (see Proposition 2.7). From this point of view, the equivariant higher spectral flow here is the same as the term $\mathrm{Ind}((\mathcal{E} \times I)_{bt})$ in [18, §2.5.8], which is studied using the KK-theory there. This enable us to replace the techniques of KK-theory in [18] by that of equivariant higher spectral flow, which is purely analytic.

Let D be an equivariant B-family on $\mathcal{F}$. A perturbation operator with respect to D is a family of bounded pseudodifferential operators A such that $D + A$ is an invertible equivariant B-family on $\mathcal{F}$, which is a generalization of A_P. Note that if at least one component of the fibers of $\mathcal{F}$ has nonzero dimension, a perturbation operator exists with respect to D if and only if $\mathrm{Ind}\, D = 0 \in K_G^*(B)$.

If the G-action on B is trivial, for any $g \in G$, we define the equivariant eta form $\tilde{\eta}_g(\mathcal{F}, A) \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C})$ with respect to a perturbation operator A in Definition 3.12. If the equivariant geometric families $\mathcal{F}$ and $\mathcal{F}'$ have the same topological structure, we prove the anomaly formula as follows.

Theorem 1.2. *Assume that the G-action on B is trivial. Let $\mathcal{F}$, $\mathcal{F}' \in \mathrm{F}^*_G(B)$ which have the same topological structure. Let A, A' be perturbation operators with respect to $D(\mathcal{F})$, $D(\mathcal{F}')$ and P, P' be the APS projections onto the eigenspaces of the positive spectrum of $D(\mathcal{F}) + A$, $D(\mathcal{F}') + A'$ respectively. For any $g \in G$, modulo exact forms on B, we have*

$$\begin{aligned} \tilde{\eta}_g(\mathcal{F}', A') - \tilde{\eta}_g(\mathcal{F}, A) = &\int_{Z^g} \widetilde{\mathrm{Td}}_g(\nabla^{TZ}, \nabla^{L_Z}, \nabla'^{TZ}, \nabla'^{L_Z}) \,\mathrm{ch}_g(E, \nabla^E) \\ &+ \int_{Z^g} \mathrm{Td}_g(\nabla'^{TZ}, \nabla'^{L_Z})\, \widetilde{\mathrm{ch}}_g(\nabla^E, \nabla'^E) \\ &+ \mathrm{ch}_g\left(\mathrm{sf}_G\{(D(\mathcal{F}) + A, P), (D(\mathcal{F}') + A', P')\}\right), \end{aligned} \tag{1.2}$$

where Z^g is the fixed point set of g on the fibers Z and the characteristic forms $\mathrm{ch}_g(\cdot)$, $\mathrm{Td}_g(\cdot)$ and the Chern-Simons forms $\widetilde{\mathrm{ch}}_g(\cdot)$, $\widetilde{\mathrm{Td}}_g(\cdot)$ are defined in Section 2.

Note that when $\mathcal{F}$, $\mathcal{F}' \in \mathrm{F}^0_G(B)$, the proof of the anomaly formula relies on a special case of functoriality of equivariant eta forms.

If $B = \mathrm{pt}$, $\mathcal{F} \in \mathrm{F}^1_G(\mathrm{pt})$, taking $A = P_{\ker D}$, the orthogonal projection onto the kernel of $D(\mathcal{F})$, the equivariant eta form here is just the classical reduced equivariant APS eta invariant. Using Theorem 1.2, we could write the equivariant spectral flow term in the anomaly formula of eta invariants explicitly.

Let $\pi : V \to B$ be an equivariant surjective proper submersion with compact orientable fibers Y. We assume that B is compact, G acts trivially on B and TY is equivariant Spin^c. Let $\mathcal{F}_X = (W, L_X, E, o_X, T^H_{\pi_X} W, g^{TX}, h^{L_X}, \nabla^{L_X}, h^E, \nabla^E)$ be an equivariant geometric family over V for an equivariant surjective proper submersion $\pi_X : W \to V$ with compact orientable fibers X (see (2.33)). Then $\pi_Z := \pi_Y \circ \pi_X : W \to B$ is an equivariant proper submersion with compact orientable fibers Z. We could obtain a new equivariant geometric family $\mathcal{F}_Z$ over B in (2.37). For any $g \in G$, let Y^g and Z^g be the fixed point sets of g on the fibers Y and Z respectively. We obtain the functoriality of equivariant eta forms.

Theorem 1.3. *Let A_Z and A_X be perturbation operators with respect to $D(\mathcal{F}_Z)$ and $D(\mathcal{F}_X)$. Then there exists $T_0 \geqslant 1$, such that for any $T \geqslant T_0$ and any $g \in G$, modulo exact forms on B, we have*

$$\begin{aligned} \widetilde{\eta}_g(\mathcal{F}_Z, A_Z) = &\int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y})\, \widetilde{\eta}_g(\mathcal{F}_X, A_X) \\ &+ \int_{Z^g} \widetilde{\mathrm{Td}}_g(\nabla^{TY,TX}, \nabla^{L_Z}, \nabla^{TZ}, \nabla^{L_Z}) \,\mathrm{ch}_g(E, \nabla^E) \\ &+ \mathrm{ch}_g(\mathrm{sf}_G\{(D(\mathcal{F}_{Z,T}) + 1\widehat{\otimes} T A_X, P), (D(\mathcal{F}_Z) + A_Z, P')\}), \end{aligned} \tag{1.3}$$

where $\mathcal{F}_{Z,T}$ is the equivariant geometric family defined in (3.9), $\nabla^{TY,TX}$ is defined in (3.91) and P, P' are the associated APS projections respectively.

In the last section, inspired by [16, 18, 46], we use the results above to define the equivariant differential K-theory for the compact manifolds when G acts with finite stabilizers and study the properties of it.

Essential to our definition is that when the group action has finite stabilizers, $K^*_G(B) \otimes \mathbb{R}$ is isomorphic to the delocalized cohomology $H^*_{deloc,G}(B, \mathbb{R})$ defined in (4.8), which is the cohomology of complex $(\Omega^*_{deloc,G}(B, \mathbb{R}), d)$ of differential forms on the disjoint union of the fixed point set of a representative element in the conjugacy classes. Furthermore, we could define $\widetilde{\eta}_G(\mathcal{F},A) \in \Omega^*_{deloc,G}(B,\mathbb{R})/\mathrm{Im} d$ (see Definition 4.2) when G acts with finite stabilizers on B.

A cycle for an equivariant differential K-theory class over B is a pair $(\mathcal{F},\rho)$, where $\mathcal{F} \in \mathrm{F}^*_G(B)$ and $\rho \in \Omega^*_{deloc,G}(B,\mathbb{R})/\mathrm{Im}\, d$. The cycle $(\mathcal{F},\rho)$ is called even (resp. odd) if $\mathcal{F}$ is even (resp. odd) and $\rho \in \Omega^{\mathrm{odd}}_{deloc,G}(B,\mathbb{R})/\mathrm{Im}\, d$ (resp. $\rho \in \Omega^{\mathrm{even}}_{deloc,G}(B,\mathbb{R})/\mathrm{Im}\, d$). Two cycles $(\mathcal{F},\rho)$ and $(\mathcal{F}',\rho')$ are called isomorphic if $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic and $\rho=\rho'$. Let $\widehat{\mathrm{IC}}^0_G(B)$ (resp. $\widehat{\mathrm{IC}}^1_G(B)$) denote the set of isomorphism classes of even (resp. odd) cycles over B. Let $\mathcal{F}^{\mathrm{op}}$ be the equivariant geometric family reversing the $\mathbb{Z}_2$-grading of E in $\mathcal{F}$, which implies that $\mathrm{Ind}(D(\mathcal{F}^{\mathrm{op}})) = -\,\mathrm{Ind}(D(\mathcal{F}))$. We call two cycles $(\mathcal{F},\rho)$ and $(\mathcal{F}',\rho')$ paired if

$$\mathrm{Ind}(D(\mathcal{F})) = \mathrm{Ind}(D(\mathcal{F}')), \tag{1.4}$$

and there exists a perturbation operator A with respect to $D(\mathcal{F}+\mathcal{F}'^{\mathrm{op}})$ such that

$$\rho-\rho' = \widetilde{\eta}_G(\mathcal{F}+\mathcal{F}'^{\mathrm{op}},A). \tag{1.5}$$

Note that from (1.4), $\mathrm{Ind}(\mathcal{F}+\mathcal{F}'^{\mathrm{op}}) = 0 \in K^*_G(B)$. So $\widetilde{\eta}_G(\mathcal{F}+\mathcal{F}'^{\mathrm{op}},A)$ is well defined. Let $\sim$ denote the equivalence relation generated by the relation "paired".

Definition 1.4. (Compare with [16, Definition 2.14]) The equivariant differential K-theory $\widehat{K}^0_G(B)$ (resp. $\widehat{K}^1_G(B)$) is the group completion of the abelian semigroup $\widehat{\mathrm{IC}}^0_G(B)/\sim$ (resp. $\widehat{\mathrm{IC}}^1_G(B)/\sim$).

Let $\pi_Y : V \to B$ be an equivariant proper submersion of compact smooth G-manifolds with compact fibers Y such that TY is oriented and equivariant Spin^c. We assume that the G-action on B has only finite stabilizers. Thus, so is the action on V. As in [16], we define the equivariant differential K-orientation with respect to π_Y in Definition 4.6 and the map $\widehat{\pi}_Y! : \widehat{\mathrm{IC}}^*_G(V) \to \widehat{\mathrm{IC}}^*_G(B)$ in (4.24). Then using Theorems 1.2 and 1.3, we prove that

Theorem 1.5. *The map $\widehat{\pi}_Y! : \widehat{K}^*_G(V) \to \widehat{K}^*_G(B)$ is well-defined.*

By Theorems 1.2 and 1.3, in Section 3, we also prove that our model is a ring valued functor with the usual properties of the differential extension of a generalized cohomology. Finally, we explain that this model could be naturally regarded as a model of differential K-theory for orbifolds.

Note that there is no adiabatic limit in Theorem 1.3. So in non-equivariant case, our proofs in the construction of the differential K-theory simplify that in [16] a little.

This paper is organized as follows.

In Section 1, we give a geometric description of the equivariant K-theory. In Section 2, we extend the spectral section to the equivariant case, introduce the equivariant higher spectral flow for arbitrary dimensional fibers and use them to obtain the anomaly formula and the functoriality of the equivariant eta forms. In Section 3, we construct an analytic model for the equivariant differential K-theory and prove some properties.

Notation: All manifolds in this paper are smooth and without boundary. We denote by d the exterior differential operator and d^B when we like to insist the base manifold B.

We use the Einstein summation convention in this paper: when an index variable appears twice in a single term and is not otherwise defined, it implies summation of that term over all the values of the index.

We also use the superconnection formalism of Quillen [47] and Bismut-Cheeger [9]. If $\mathcal{A}$ is a $\mathbb{Z}_2$-graded algebra, and if $a, b \in \mathcal{A}$, then we will note $[a, b]$ as the supercommutator of a, b. In the whole paper, if E, E' are two $\mathbb{Z}_2$-graded spaces we will note $E\widehat{\otimes}E'$ as the $\mathbb{Z}_2$-graded tensor product as in [6, §1.3]. If one of E, E' is ungraded, we understand it as $\mathbb{Z}_2$-graded by taking its odd part as zero. Let P be a trace class operator on a $\mathbb{Z}_2$-graded space $E = E_+ \oplus E_-$. Let $P|_{E_+}$ and $P|_{E_-}$ be the restrictions of P on E_+ and E_- respectively. We denote by

$$\mathrm{Tr}_s[P] = \mathrm{Tr}[P|_{E_+}] - \mathrm{Tr}[P|_{E_-}]. \tag{1.6}$$

For the fiber bundle $\pi : W \to B$, we will often use the integration of the differential forms along the oriented fibres Z in this paper. Since the fibers may be odd dimensional, we must make precisely our sign conventions: for $\alpha \in \Omega^\bullet(B)$ and $\beta \in \Omega^\bullet(W)$, then

$$\int_Z (\pi^*\alpha) \wedge \beta = \alpha \wedge \int_Z \beta. \tag{1.7}$$

2 Equivariant K-theory

In this section, we explain a geometric description of the equivariant K-theory in [16,18] for any compact Lie group action and define the push-forward map of equivariant K-groups in this point of view. In Section 2.1, we recall some elementary results of the Clifford algebra. In Section 2.2, we introduce the equivariant geometric family. In Section 2.3, we give a geometric description of the equivariant K-theory. In Section 2.4, we study the push-forward map in equivariant K-theory using equivariant geometric families.

2.1 Clifford algebra

Let E be an oriented Euclidean space of dimension n. Let $C(E)$ denote the complex Clifford algebra of E. Relative to an orthonormal basis of E, $\{e_i\}_{1\leqslant i\leqslant n}$, $C(E)$ is defined by the relations

$$e_ie_j + e_je_i = -2\delta_{ij}. \tag{2.1}$$

To avoid ambiguity, we denote by $c(e_i)$ the element of $C(E)$ corresponding to e_i. The Clifford algebra $C(E)$ is naturally $\mathbb{Z}_2$-graded from the $\mathbb{N}$-grading of the tensor algebra after reduction mod 2. We denote by $C(E) = C_0(E) \oplus C_1(E)$. Let Spin_n^c be the Spin^c group associated with $C(E)$ (cf. [32, Appendix D]).

If $n = 2k$ is even, up to isomorphism, $C(E)$ has a unique irreducible representation, the spinor $\mathcal{S}(E)$, which has a $\mathbb{Z}_2$-grading obtained from the chirality operator

$$\tau_E = (\sqrt{-1})^k c(e_1) \cdots c(e_{2k}).$$

We write $\mathcal{S}(E) = \mathcal{S}_+(E) \oplus \mathcal{S}_-(E)$ with respect to this $\mathbb{Z}_2$-grading. In fact, there are isomorphisms of $\mathbb{Z}_2$-graded algebras

$$C(E) \simeq \mathrm{End}(\mathcal{S}(E)) \simeq \mathcal{S}(E)\widehat{\otimes}\mathcal{S}(E)^*. \tag{2.2}$$

For any $A \in C(E)$, we write $\mathrm{Tr}_s[A] := \mathrm{Tr}[\tau_E A]$ the supertrace on $\mathcal{S}(E)$. Note that $\mathcal{S}(E)$ is also a representation of Spin_n^c induced by the Clifford action.

If $n = 2k - 1$ is odd, $C(E)$ has only two inequivalent irreducible representations. For arbitrary n, $c(e_j) \mapsto c(e_j)c(e_{n+1})$, $1 \leqslant j \leqslant n$, defines an isomorphism $C(E) \simeq$

$C_0(E\oplus\mathbb{R})$ of algebras. Since n is odd, we can regard $\mathcal{S}_\pm(E\oplus\mathbb{R})$ as the two inequivalent irreducible representations of $C(E)$. Their restrictions to Spin_n^c are equivalent. In the following, we may and we take $\mathcal{S}_+(E\oplus\mathbb{R})$ as the spinor for $C(E)$, also denoted by $\mathcal{S}(E)$ for the convenience. In particular, the notation $\mathrm{Tr}[\cdot]$ on the spinor refers to the representation $\mathcal{S}_+(E\oplus\mathbb{R})$.

Let F be another oriented Euclidean space. Let $\mathcal{S}(E)\widehat{\otimes}\mathcal{S}(F)$ be the $\mathbb{Z}_2$-graded tensor product of $\mathcal{S}(E)$ and $\mathcal{S}(F)$. Then it is a $\mathbb{Z}_2$-graded representation of $C(E)\widehat{\otimes}C(F)$ defined by

$$(a_1\widehat{\otimes}a_2)(s_1\widehat{\otimes}s_2)=(-1)^{|a_2|\cdot|s_1|}(a_1s_1)\widehat{\otimes}(a_2s_2), \tag{2.3}$$

where $a_1\in C(E)$, $a_2\in C(F)$, $s_1\in\mathcal{S}(E)$, $s_2\in\mathcal{S}(F)$ and $|a_2|$, $|s_1|$ are degrees of a_2, s_1 associated with the $\mathbb{Z}_2$-gradings of $C(F)$, $\mathcal{S}(E)$ respectively. We express $\mathcal{S}(E)\widehat{\otimes}\mathcal{S}(F)$ by $\mathcal{S}(E)$ and $\mathcal{S}(F)$ using ungraded tensor product as follows (cf. [9, (1.10), (1.11)]).

If both $\dim E$ and $\dim F$ are odd, let $\mathbb{C}^2=\mathbb{C}\oplus\mathbb{C}$ define the grading on $\mathbb{C}^2$ and let $J,K\in\mathrm{End}(\mathbb{C}^2)$ denote the involutions

$$J=\begin{pmatrix}0 & -\sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix},\quad K=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{2.4}$$

Note that $J^2=K^2=1$, $JK=-KJ$. Then $\mathcal{S}(E)\otimes\mathcal{S}(F)\otimes\mathbb{C}^2$ with involution $1\otimes1\otimes\sqrt{-1}JK$ is the unique irreducible $\mathbb{Z}_2$-graded representation of $C(E)\widehat{\otimes}C(F)$ defined by

$$a_i\widehat{\otimes}b_j\to a_i\otimes b_j\otimes J^iK^j,\quad a_i\in C_i(E), b_j\in C_j(F). \tag{2.5}$$

It is isomorphic to $\mathcal{S}(E)\widehat{\otimes}\mathcal{S}(F)$ as $\mathbb{Z}_2$-graded $C(E)\widehat{\otimes}C(F)$-representations.

If $\dim E$ is even and $\dim F$ is odd, then as representations, $\mathcal{S}(E)\widehat{\otimes}\mathcal{S}(F)$ is isomorphic to $\mathcal{S}(E)\otimes\mathcal{S}(F)$ with $C(E)\widehat{\otimes}C(F)$-action defined by

$$a\widehat{\otimes}b_i\to a\tau_E^i\otimes b_i,\quad a\in C(E), b_i\in C_i(F). \tag{2.6}$$

If $\dim E$ is odd and $\dim F$ is even, then as representations, $\mathcal{S}(E)\widehat{\otimes}\mathcal{S}(F)$ is isomorphic to $\mathcal{S}(E)\otimes\mathcal{S}(F)$ with $C(E)\widehat{\otimes}C(F)$-action defined by

$$a_i\widehat{\otimes}b\to a_i\otimes\tau_F^ib,\quad a_i\in C_i(E), b\in C(F). \tag{2.7}$$

If both $\dim E$ and $\dim F$ are even, the representation $\mathcal{S}(E)\otimes\mathcal{S}(F)$ with $C(E)\widehat{\otimes}C(F)$-action defined by (2.7) is the unique irreducible one and $\mathbb{Z}_2$-graded for the tensor product grading on $\mathcal{S}(E)\otimes\mathcal{S}(F)$. It is isomorphic to $\mathcal{S}(E)\widehat{\otimes}\mathcal{S}(F)$ as representations.

Since

$$C(E\oplus F)\simeq C(E)\widehat{\otimes}C(F), \tag{2.8}$$

$\mathcal{S}(E)\widehat{\otimes}\mathcal{S}(F)$ is also a $\mathbb{Z}_2$-graded representation of $C(E\oplus F)$. By (2.8), we have the isomorphism of representations

$$\mathcal{S}(E\oplus F)\simeq\mathcal{S}(E)\widehat{\otimes}\mathcal{S}(F). \tag{2.9}$$

2.2 Equivariant geometric family

In this subsection, we introduce the equivariant geometric family (cf. [16, 18]).

Let $\pi: W\to B$ be a smooth surjective proper submersion of compact manifolds with compact fibers Z (possibly non-connected). Let $B=\sqcup_iB_i$ be a finite disjoint union of compact connected manifolds. Let W_i be the restriction of W on B_i. Let $W_i = \sqcup_j W_{ij}$ be a finite disjoint union of compact connected manifolds. Let Z_{ij} be the fibers of the submersions restricted on W_{ij}. We note here that the dimension of Z_{ij} might be zero. In the sequel, we will often omit the subscripts i, j.

Let $TZ = \ker(d\pi)$ be the relative tangent bundle to the fibers Z over W, which is a subbundle of TW. We assume that TZ is orientable and carries an orientation $o \in H^0(W, \mathbb{Z}_2)$. Let $T_\pi^H W$ be a horizontal subbundle of TW such that

$$TW = T_\pi^H W \oplus TZ. \tag{2.10}$$

The splitting (2.10) gives an identification

$$T_\pi^H W \cong \pi^* TB. \tag{2.11}$$

If there is no ambiguity, we will omit the subscript π in $T_\pi^H W$. Let P^{TZ} be the projection

$$P^{TZ} : TW = T^H W \oplus TZ \to TZ. \tag{2.12}$$

Let g^{TZ}, g^{TB} be Riemannian metrics on TZ, TB. We equip $TW = T^H W \oplus TZ$ with the Riemannian metric

$$g^{TW} = \pi^* g^{TB} \oplus g^{TZ}. \tag{2.13}$$

Let ∇^{TW} be the Levi-Civita connection on (W, g^{TW}). Set

$$\nabla^{TZ} = P^{TZ} \nabla^{TW} P^{TZ}. \tag{2.14}$$

Then ∇^{TZ} is a Euclidean connection on TZ. By [8, Theorem 1.9], we know that ∇^{TZ} only depends on $(T^H W, g^{TZ})$.

Let $C(TZ)$ be the Clifford algebra bundle of (TZ, g^{TZ}), whose fiber at $x \in W$ is the Clifford algebra $C(T_x Z)$ of the Euclidean space $(T_x Z, g^{T_x Z})$.

We make the assumption that the oriented vector bundle (TZ, o) **has a Spin**^{c} **structure.** Then there exists a complex line bundle L_Z over W such that $\omega_2(TZ) = c_1(L_Z) \mod 2$, where ω_2 denotes the second Stiefel-Whitney class and c_1 denotes the first Chern class. Let $\mathcal{S}(TZ, L_Z)$ be the fundamental complex spinor bundle for (TZ, L_Z), which has a smooth action of $C(TZ)$ (cf. [32, Appendix D.9]). Locally, the spinor bundle $\mathcal{S}(TZ, L_Z)$ may be written as

$$\mathcal{S}(TZ, L_Z) = \mathcal{S}(TZ) \otimes L_Z^{1/2}, \tag{2.15}$$

where $\mathcal{S}(TZ)$ is the fundamental spinor bundle for the (possibly non-existent) spin structure on TZ, and $L_Z^{1/2}$ is the (possibly non-existent) square root of L_Z. Let h^{L_Z} be a Hermitian metric on L_Z. Then from (2.15), the metrics g^{TZ} and h^{L_Z} induce a Hermitian metric on $\mathcal{S}(TZ, L_Z)$, which we denote by $h^{\mathcal{S}_Z}$ for simplicity. Let ∇^{L_Z} be a Hermitian connection on (L_Z, h^{L_Z}). Similarly, we denote by $\nabla^{\mathcal{S}_Z}$ the connection on $\mathcal{S}(TZ, L_Z)$ induced by ∇^{TZ} and ∇^L from (2.15). Then $\nabla^{\mathcal{S}_Z}$ is a Hermitian connection on $(\mathcal{S}(TZ, L_Z), h^{\mathcal{S}_Z})$. Moreover, it is a Clifford connection associated with ∇^{TZ}, i.e., for any $U \in TW$, $V \in \mathcal{C}^\infty(W, TZ)$,

$$\left[\nabla_U^{\mathcal{S}_Z}, c(V)\right] = c\left(\nabla_U^{TZ} V\right). \tag{2.16}$$

In the following, we often simply denote the spinor bundle $\mathcal{S}(TZ, L_Z)$ by $\mathcal{S}_Z$. If $n = \dim Z$ is even, $\mathcal{S}_Z$ is $\mathbb{Z}_2$-graded and the action of TZ exchanges the $\mathbb{Z}_2$-grading.

Let $E = E_+ \oplus E_-$ be a $\mathbb{Z}_2$-graded smooth complex vector bundle over W with Hermitian metric h^E, for which E_+ and E_- are orthogonal, and let ∇^E be a Hermitian connection on (E, h^E) preserving the $\mathbb{Z}_2$-grading. Set

$$\nabla^{\mathcal{S}_Z \widehat{\otimes} E} := \nabla^{\mathcal{S}_Z} \widehat{\otimes} 1 + 1 \widehat{\otimes} \nabla^E. \tag{2.17}$$

Then $\nabla^{\mathcal{S}_Z \widehat{\otimes} E}$ is a Hermitian connection on $(\mathcal{S}_Z \widehat{\otimes} E, h^{\mathcal{S}_Z} \otimes h^E)$.

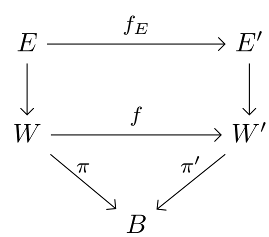
Let G be a compact Lie group which acts on W and B such that for any $g \in G$, $\pi \circ g = g \circ \pi$. We assume that the G-action preserves the splitting (2.10) and the Spin^c structure of TZ. Thus TZ, L_Z, $\mathcal{S}_Z$ are G-equivariant vector bundles. We assume that g^{TZ}, h^{L_Z}, ∇^{L_Z} are G-invariant. We further assume that E is a G-equivariant $\mathbb{Z}_2$-graded complex vector bundle and h^E, ∇^E are G-invariant. Note that the G-action here may be nontrivial on B.

Definition 2.1. (Compare with [16, Definition 2.2]) An equivariant geometric family $\mathcal{F}$ over B is a family of G-equivariant geometric data

$$\mathcal{F} = (W, L_Z, E, o, T^H W, g^{TZ}, h^{L_Z}, \nabla^{L_Z}, h^E, \nabla^E) \tag{2.18}$$

described above. We call the equivariant geometric family $\mathcal{F}$ is even (resp. odd) if for any connected component of fibers, the dimension is even (resp. odd).

Definition 2.2. (Compare with [16, §2.1.7]) Let $\mathcal{F}$ and $\mathcal{F}'$ be two equivariant geometric families over B. An isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{F}'$ consists of the following data:



where

1. f is a diffeomorphism commuting with the G-action such that $\pi' \circ f = \pi$, which implies that f preserves the relative tangent bundle;
2. f preserves the orientation and the Spin^c structure of the relative tangent bundle, which implies that there exists an equivariant complex line bundle isomorphism $f_L : L_Z \to L'_Z$;
3. $f_E : E \to E'$ is an equivariant vector bundle isomorphism over f, which preserves the $\mathbb{Z}_2$-grading;
4. f preserves the horizontal subbundle and the vertical metric;
5. f_L and f_E preserve the metrics and the connections on the vector bundles.

If only the first three conditions hold, we say that $\mathcal{F}$ and $\mathcal{F}'$ have **the same topological structure**.

Let $\mathrm{F}^0_G(B)$ (resp. $\mathrm{F}^1_G(B)$) be the set of equivalence classes of even (resp. odd) equivariant geometric families over B.

For two equivariant geometric families $\mathcal{F}, \mathcal{F}'$ over B, we can form their sum $\mathcal{F}+\mathcal{F}'$ over B as a new equivariant geometric family: the underlying fibration of the sum is $\pi \sqcup \pi' : W \sqcup W' \to B$, where $\sqcup$ is the disjoint union and the remaining structures of $\mathcal{F} + \mathcal{F}'$ are induced in the obvious way. Let $\mathrm{F}^*_G(B) = \mathrm{F}^0_G(B) \oplus \mathrm{F}^1_G(B)$. It is an additive abelian semigroup.

For $\mathcal{F}, \mathcal{F}' \in \mathrm{F}^*_G(B)$, we can also form their product $\mathcal{F} \times_B \mathcal{F}'$ over B. The total space of the underlying fibration of $\mathcal{F} \times_B \mathcal{F}'$ is $W \times_B W' := \{(w, w') \in$

$W \times W' : \pi(w) = \pi'(w')\}$ and the fiber is $Z \times Z'$. Let $\mathrm{pr}_W : W \times_B W' \to W$ and $\mathrm{pr}_{W'} : W \times_B W' \to W'$ be the obvious projections. The complex vector bundle now is $\mathrm{pr}_W^* E \widehat{\otimes} \mathrm{pr}_{W'}^* E'$. The remaining structures of $\mathcal{F} \times_B \mathcal{F}'$ are induced in the obvious way.

Let B, B' be two compact manifolds with smooth G-action. Let $f : B \times B' \to B$ be the projection onto the first part. For any $\mathcal{F} \in \mathrm{F}_G^*(B)$, we could construct the pullback $f^*\mathcal{F} \in \mathrm{F}_G^*(B \times B')$ in a natural way. Remark that in general case, for a G-equivariant map $f : B' \to B$ and $\mathcal{F} \in \mathrm{F}_G^*(B)$, $f^*\mathcal{F}$ is hard to define canonically because we cannot choose a canonical horizontal subbundle in $f^*\mathcal{F}$. We will show more details in Section 4.3 later.

Definition 2.3. The opposite family $\mathcal{F}^{\mathrm{op}}$ of an equivariant geometric family $\mathcal{F}$ is obtained by reversing the $\mathbb{Z}_2$-grading of E.

2.3 Equivariant K-Theory

In this subsection, we give some examples of the equivariant geometric families and a geometric description of the equivariant K-theory.

Let $K_G^0(B)$ be the G-equivariant K^0-group of B, which is the Grothendieck group of the equivalence classes of G-equivariant topological complex vector bundles over B (cf. [48]). Since G is compact, by Proposition A.4, it is also the Grothendieck group of the equivalence classes of G-equivariant smooth complex vector bundles. Note that the ring structure of the K^0-group is induced by the tensor product of the complex vector bundles.

We lift the G-action on $B \times S^1$ such that the G-action on S^1 is trivial. Take $s \in S^1$ fixed. Let $i : B \ni b \to (b, s) \in B \times S^1$ be the G-equivariant inclusion map. Let $i^* : K_G^0(B \times S^1) \to K_G^0(B)$ be the induced homomorphism. Let $K_G^1(B)$ be the G-equivariant K^1-group of B. By [48, Definitions 2.7 and 2.8], we have the split short exact sequence

$$0 \longrightarrow K_G^1(B) \overset{j}{\longrightarrow} K_G^0(B \times S^1) \overset{i^*}{\longrightarrow} K_G^0(B) \longrightarrow 0, \tag{2.19}$$

where j is induced by the suspension isomorphism $K_G^1(B) \simeq \widetilde{K}_G^0(B \wedge S^1) \simeq \ker(i^*)$ (cf. [48, p136]). Here $B \wedge S^1$ is the smash product of B and S^1 and $\widetilde{K}_G^0(B \wedge S^1)$ is the G-equivariant reduced K^0-group of $B \wedge S^1$.

Now we introduce another explanation of $K_G^1(B)$. Let V be a finite dimensional complex unitary representation of G. If $F \in \mathcal{C}^\infty(B, \mathrm{End}(V))$ such that for any $b \in B$, $F(b) \in \mathrm{End}(V)$ is unitary and for any $g \in G$, $v \in V$,

$$g(F(b)v) = F(gb)(gv), \tag{2.20}$$

we say F is a G-invariant unitary element of $\mathcal{C}^\infty(B, \mathrm{End}(V))$. In this case, for $(b, t, v) \in B \times [0, 1] \times V$, the relation $(b, 0, v) \sim (b, 1, F(b)v)$ forms a G-equivariant smooth Hermitian vector bundle W over $B \times S^1$. Let $U = B \times S^1 \times V$ be the G-equivariant trivial bundle over $B \times S^1$ as in (A.2). Then from (2.19), $[W] - [U] \in \ker(i^*)$ corresponds to an element $[F] \in K_G^1(B)$.

Lemma 2.4. *For any $y \in K_G^1(B)$, there exists a finite dimensional complex unitary representation V of G, such that y can be represented as a G-invariant unitary element of $\mathcal{C}^\infty(B, \mathrm{End}(V))$.*

Proof. By (2.19), an element $y \in K_G^1(B)$ can be represented as an element $x = j(y) \in K_G^0(B \times S^1)$ such that $i^*x = 0 \in K_G^0(B)$. We write $x = W - U$, where W and U are equivariant smooth complex vector bundles over $B \times S^1$. By Proposition A.2, we may and we will assume that U is an equivariant trivial complex vector bundle over $B \times S^1$ associated with a finite dimensional complex G-representation V as in (A.2). Note that $B \times S^1 \simeq B \times \mathbb{R}/\mathbb{Z}$. We assume that $i(B) = B \times \{1/2\}$. Since $i^*x = W|_{B\times\{1/2\}} - U|_{B\times\{1/2\}} = 0 \in K_G^0(B)$, by Proposition A.2, we may and we will assume that $W|_{B\times\{1/2\}}$ is isomorphic to the equivariant trivial bundle $(B\times\{1/2\})\times V$ over $B\times\{1/2\}$ as equivariant smooth complex vector bundles. Since $(0,1)$ is contractible, as equivariant smooth complex vector bundles over $B \times (0,1)$, $W|_{B\times(0,1)} \simeq (\mathrm{Id}_B \times p_{1/2})^*(W|_{B\times\{1/2\}})$ where $p_{1/2} : (0,1) \to 1/2$ is the constant map. Since $B \times (0,1) \times V = (\mathrm{Id}_B \times p_{1/2})^*((B \times \{1/2\}) \times V)$ as complex vector bundles over $B \times (0,1)$, there exists a G-equivariant smooth complex vector bundle isomorphism

$$f : W|_{B\times(0,1)} \to B \times (0,1) \times V. \tag{2.21}$$

Let $h : B \times (0,1) \times V \to V$ be the obvious projection. For any $b \in B$, $v \in V$, we could choose a section $s \in \mathcal{C}^\infty(B \times S^1, W)$ such that $\lim_{t\to 0} h \circ f(s(b,t)) = v$. Then we define

$$F(b)v := \lim_{t\to 1} h \circ f(s(b,t)) \in V. \tag{2.22}$$

Note that the definition of $F(b) \in \mathrm{End}(V)$ does not depend on the choices of the isomorphic map f and the section s. Take a G-invariant Hermitian metric on W which induces a G-invariant Hermitian inner product on V. It is obvious that F is a G-invariant unitary element of $\mathcal{C}^\infty(B, \mathrm{End}(V))$ and $[F] = y \in K_G^1(B)$.

The proof of Lemma 2.4 is completed. □

For an equivariant geometric family $\mathcal{F}$, the fiberwise Dirac operator $D(\mathcal{F})$ associated with $\mathcal{F}$ is defined by

$$D(\mathcal{F}) := \sum_i c(e_i)\nabla_{e_i}^{\mathcal{S}_Z\widehat{\otimes}E}, \tag{2.23}$$

where $\{e_i\}$ is a local orthonormal frame of TZ. Note that the definition of the fiberwise Dirac operator is independent of the choice of the local orthonormal frame. From (2.23), the G-action commutes with $D(\mathcal{F})$. If $\mathcal{F}$ is isomorphic to $\mathcal{F}'$, from Definition 2.2 and (2.23), the isomorphism preserves the fiberwise Dirac operator. So the fiberwise Dirac operator can be defined on an element of $\mathrm{F}_G^*(B)$. For an even (resp. odd) equivariant geometric family $\mathcal{F}$, the classical construction of Atiyah-Singer assigns to this family its equivariant (analytic) index $\mathrm{Ind}(D(\mathcal{F})) \in K_G^0(B)$ (resp. $K_G^1(B)$) (cf. [4,5]). Remark that $\mathrm{Ind}(D(\mathcal{F}))$ depends only on the topological structure of $\mathcal{F}$. It induces a map

$$\begin{aligned} \mathrm{Ind} : \mathrm{F}_G^*(B) &\to K_G^*(B), \\ \mathcal{F} &\mapsto \mathrm{Ind}(D(\mathcal{F})). \end{aligned} \tag{2.24}$$

Let $K_G^*(B) = K_G^0(B) \oplus K_G^1(B)$. Since

$$\mathrm{Ind}(D(\mathcal{F} + \mathcal{F}')) = \mathrm{Ind}(D(\mathcal{F})) + \mathrm{Ind}(D(\mathcal{F}')) \in K_G^*(B), \tag{2.25}$$

the equivariant index map in (2.24) is a semigroup homomorphism. It is well-known that if $\mathcal{F}$ and $\mathcal{F}'$ are even,

$$\mathrm{Ind}(D(\mathcal{F} \times_B \mathcal{F}')) = \mathrm{Ind}(D(\mathcal{F})) \cdot \mathrm{Ind}(D(\mathcal{F}')) \in K_G^0(B). \tag{2.26}$$

Example 2.5. a) Let (E, h^E) be an equivariant $\mathbb{Z}_2$-graded smooth Hermitian vector bundle over B with a G-invariant Hermitian connection ∇^E. Then (E, h^E, ∇^E)

can be regarded as an even equivariant geometric family $\mathcal{F}$ for $Z = \mathrm{pt}$. In this case, $D(\mathcal{F}) = 0$ and $\mathrm{Ind}(D(\mathcal{F})) = [E_+] - [E_-] \in K_G^0(B)$.

b) Let $W = B \times \mathbb{C}P^1$ with G-action which acts trivially on $\mathbb{C}P^1$. Then the complex line bundle $\mathcal{O}(1)$ over $\mathbb{C}P^1$ can be naturally extended on W. Thus $(W, \mathcal{O}(1))$ with canonical metrics, connections, the standard orientation o of $\mathbb{C}P^1$ and the Spin structure on $\mathbb{C}P^1$ form an even equivariant geometric family $\mathcal{F}_S$ over B. Let $D^{\mathcal{O}(1)}_{\mathbb{C}P^1}$ be the Dirac operator on $\mathbb{C}P^1$ associated with $\mathcal{O}(1)$. Since $\mathrm{Ind}(D^{\mathcal{O}(1)}_{\mathbb{C}P^1}) = \langle c_1(\mathcal{O}(1)), [\mathbb{C}P^1]\rangle = 1$, from (2.26), for even equivariant geometric family $\mathcal{F}$ in a), we have $\mathrm{Ind}(D(\mathcal{F} \times_B \mathcal{F}_S)) = \mathrm{Ind}(D(\mathcal{F})) \in K_G^0(B)$.

c) (Compare with [44, §5] and [14, §2.2.3.8]) Let $B = S^1_\theta = \mathbb{R}/\mathbb{Z}$, $W = S^1_\theta \times S^1_t$ and $\pi : W \to B$ be the projection onto the first part. We consider the Hermitian line bundle (L, h^L) which is obtained by identifying

$$(\theta = 0, t, v),\ (\theta = 1, t, \exp(-2\pi t\sqrt{-1})v) \in [0,1] \times S^1_t \times \mathbb{C}. \tag{2.27}$$

Then

$$\nabla^L = d + 2\pi\,(\theta - 1/2)\,\sqrt{-1}dt \tag{2.28}$$

is a Hermitian connection on (L, h^L) (cf. [11, p.124]). We choose the $\mathbb{Z}_2$-grading of L such that $L_+ = L$ and $L_- = 0$. We consider the Spin structure on S^1_t as the desired Spin^c structure. Then we get an odd geometric family $\mathcal{F}^L$ after choosing the natural geometric data. In fact, since $c_1(L) = dtd\theta$, $\mathrm{Ind}(D(\mathcal{F}^L))$ is a generator of $K^1(S^1) \simeq \mathbb{Z}$ by family index theorem.

d) Let $\mathcal{F} \in \mathrm{F}^*_G(B)$. Let p_1 and p_2 be the projections onto the first and second parts of $B \times S^1$ respectively. We take $\mathcal{F}^L$ as in c). Then $p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L$ is an equivariant geometric family over $B \times S^1$. From Proposition B.1 (cf. also the proof of [11, Theorem 2.10]), for $\mathcal{F} \in \mathrm{F}^1_G(B)$, there exists an inclusion $i : B \to B \times S^1$ such that $i^*\,\mathrm{Ind}(D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L)) = 0$. Moreover, as an element of $K^1_G(B)$ in the sense of (2.19), by an equivariant version of [44, Proposition 6], we have

$$j\big(\mathrm{Ind}(D(\mathcal{F}))\big) = \mathrm{Ind}(D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L)), \tag{2.29}$$

where j is the map in (2.19). This example is essential in our construction of the higher spectral flow for even case. For the sake of reader's convenience, we will show more details in Appendix B.

We denote by $\mathcal{F} \sim \mathcal{F}'$ if $\mathrm{Ind}(D(\mathcal{F})) = \mathrm{Ind}(D(\mathcal{F}'))$. It is an equivalence relation and compatible with the semigroup structure. So $\mathrm{F}^*_G(B)/\sim$ is a semigroup and the map

$$\mathrm{Ind} : \mathrm{F}^*_G(B)/\sim\ \longrightarrow K^*_G(B) \tag{2.30}$$

is an injective semigroup homomorphism.

By Definition 2.3, we have

$$\mathrm{Ind}(D(\mathcal{F}^{\mathrm{op}})) = -\,\mathrm{Ind}(D(\mathcal{F})). \tag{2.31}$$

After defining $-\mathcal{F} := \mathcal{F}^{\mathrm{op}}$, the semigroup $\mathrm{F}^*_G(B)/\sim$ can be regarded as an abelian group. So, by (2.31), the equivariant index map in (2.30) is a group homomorphism. Note that $K^*_G(B)$ has a ring structure [2], and by (2.29), (2.26) holds for any $\mathcal{F}, \mathcal{F}' \in \mathrm{F}^*_G(B)$. Thus the equivariant index map in (2.30) is also a ring homomorphism. In fact, it is even a ring isomorphism [18, §2.5.5]. We rewrite the proof in our notation here for the completeness.

Proposition 2.6. *The equivariant index map* Ind *in (2.30) is surjective. In other words, we have the* $\mathbb{Z}_2$*-graded ring isomorphism*

$$\mathrm{F}_G^*(B)/\sim\, \simeq K_G^*(B). \tag{2.32}$$

Proof. When $*=0$, we can get the proposition directly from Example 2.5 a) or b).

When $*=1$, from the proof of Lemma 2.4, for any $[F]\in K_G^1(B)$, there exist equivariant complex vector bundles W and U such that $[W]-[U]\in K_G^0(B\times S^1)$ corresponds to $[F]\in K_G^1(B)$. Moreover, after taking the natural geometric data, we get an odd equivariant geometric family $\mathcal{F}$ over B with fibers S^1 and $\mathbb{Z}_2$-graded equivariant complex vector bundle $W\oplus U$. As in [14, §2.2.2.3], we have $\mathrm{Ind}(D(\mathcal{F}))=[F]\in K_G^1(B)$.

The proof of Proposition 2.6 is completed. □

Remark 2.7. Note that if we replace the Spin^c condition of the geometric family by the general Clifford module condition (which is the setting in [16,18]) or the Spin condition, Proposition 2.6 also holds. Since we don't use the language of Clifford modules here, our definition of $\mathcal{F}^{\mathrm{op}}$ in Definition 2.3 is simpler than that in [16,18]. In fact, in the sense of (2.32), they are the same.

2.4 Push-forward map

In this subsection, we define the push-forward map in equivariant K-theory using the equivariant geometric families.

Let $\pi_Y: V\to B$ be a G-equivariant smooth surjective proper submersion of compact manifolds with compact orientable fibers Y. We simply assume that the dimensions of all connected components of Y have the same parity. Let $o_Y\in H^0(V,\mathbb{Z}_2)$ be an orientation of the relative tangent bundle TY.

Definition 2.8. (Compare with [18, Definition 3.1]) An equivariant K-orientation of π_Y is an equivariant Spin^c structure of TY. Let $\mathcal{O}_G(\pi_Y)$ be the set of equivariant K-orientations.

Suppose that π_Y has an equivariant K-orientation $\mathcal{O}_Y\in\mathcal{O}_G(\pi_Y)$. For $j=0,1$, let $N(j):=j$ (resp. $(j+1)\,\mathrm{mod}\,2$) if $\dim Y$ is even (resp. odd). We will use Proposition 2.6 to define the push-forward map of equivariant K-groups $\pi_Y!: K_G^j(V)\to K_G^{N(j)}(B)$ as follows.

Let $\pi_X: W\to V$ be the submersion with compact orientable fibers X. Let

$$\mathcal{F}_X=(W,L_X,E,o_X,T^H_{\pi_X}W,g^{TX},h^{L_X},\nabla^{L_X},h^E,\nabla^E) \tag{2.33}$$

be a G-equivariant geometric family in $\mathrm{F}_G^j(V)$. Then $\pi_Z:=\pi_Y\circ\pi_X: W\to B$ is a smooth submersion with compact orientable fibers Z. We have the diagram of smooth fibrations:

$$\begin{array}{ccccc}
X & \longrightarrow & Z & \longrightarrow & W & & \\
 & & \big\downarrow & & \pi_X\big\downarrow & \searrow^{\pi_Z} & \\
 & & Y & \longrightarrow & V & \xrightarrow{\ \pi_Y\ } & B.
\end{array}$$

Set $T^H_{\pi_X}Z:=T^H_{\pi_X}W\cap TZ$. Then we have the splitting of smooth vector bundles over W,

$$TZ=T^H_{\pi_X}Z\oplus TX, \tag{2.34}$$

and

$$T^H_{\pi_X}Z \cong \pi_X^* TY. \tag{2.35}$$

Let $o_Z := \pi_X^* o_Y \cup o_X \in H^0(W, \mathbb{Z}_2)$. Since TY and TX have equivariant Spin^c structures, so is TZ. Let L_Y be the equivariant complex line bundle associated with the equivariant Spin^c structure of TY. Set

$$L_Z := \pi_X^* L_Y \otimes L_X. \tag{2.36}$$

Let g^{TY} be a G-invariant Riemannian metric on TY. Let h^{L_Y} be a G-invariant Hermitian metric on L_Y and ∇^{L_Y} be a G-invariant Hermitian connection on (L_Y, h^{L_Y}). Take geometric data $(T^H_{\pi_Z}W, g^{TZ}, h^{L_Z}, \nabla^{L_Z})$ of π_Z such that $T^H_{\pi_Z}W \subset T^H_{\pi_X}W$, $g^{TZ} = \pi_X^* g^{TY} \oplus g^{TX}$, $h^{L_Z} = \pi_X^* h^{L_Y} \otimes h^{L_X}$ and $\nabla^{L_Z} = \pi_X^* \nabla^{L_Y} \otimes 1 + 1 \otimes \nabla^{L_X}$. We get a new equivariant geometric family over B,

$$\mathcal{F}_Z := (W, L_Z, E, o_Z, T^H_{\pi_Z}W, g^{TZ}, h^{L_Z}, \nabla^{L_Z}, h^E, \nabla^E). \tag{2.37}$$

We write $\mathcal{F}_Z = \pi_Y!(\mathcal{F}_X)$.

Theorem 2.9. *For equivariant K-orientation $\mathcal{O}_Y \in \mathcal{O}_G(\pi_Y)$ fixed, the push-forward map*

$$\begin{aligned} \pi_Y! : K^j_G(V) &\to K^{N(j)}_G(B), \\ [\mathcal{F}_X] &\mapsto [\mathcal{F}_Z] \end{aligned} \tag{2.38}$$

is a well-defined group homomorphism and independent of the geometric data $(T^H_{\pi_Z}W, g^{TY}, h^{L_Y}, \nabla^{L_Y})$.

Proof. We firstly assume that the map $\pi_Y!$ is well-defined. Then the remaining results follow from the definition of the equivariant family index.

The well-defined property of $\pi_Y!$ will be proved in Section 3.2 later. □

Let $\pi_U : B \to S$ be a G-equivariant smooth surjective proper submersion of compact manifolds with compact oriented fibers U and an equivariant K-orientation $\mathcal{O}_U$. Then $\pi_A := \pi_U \circ \pi_Y : V \to S$ is a G-equivariant smooth submersion with an equivariant K-orientation constructed by $\mathcal{O}_Y$ and $\mathcal{O}_U$. From the construction of the push-forward map and Theorem 2.9, the following theorem is obvious.

Theorem 2.10. *We have the equality of homomorphisms*

$$\pi_A! = \pi_U! \circ \pi_Y! : K^*_G(V) \to K^*_G(S). \tag{2.39}$$

3 Equivariant higher spectral flow and equivariant eta form

In this section, we extend the Melrose-Piazza spectral section to the equivariant case, introduce the equivariant version of Dai-Zhang higher spectral flow for arbitrary dimensional fibers and use them to prove the anomaly formula and the functoriality of the equivariant Bismut-Cheeger eta forms. In this section, we use the notation in Section 2.

In Section 3.1, we introduce the equivariant version of the spectral section and prove the main properties. In Section 3.2, we complete the proof of the well-definedness of the push-forward map in Theorem 2.9. In Section 3.3, we define the higher spectral flow for fibrations with even-dimensional fibers and extend the higher spectral flow to the equivariant case. Moreover, we prove that the equivariant K-group could be generated by the equivariant higher spectral flows. In Section 3.4, we explain the family local index theorem. In Section 3.5, we define the equivariant eta form associated with a perturbation operator. In Section 3.6, we prove the anomaly formula of equivariant eta forms in odd case. In Sections 3.7-3.9, we prove the functoriality of equivariant eta forms and use it to prove the anomaly formula in even case.

3.1 Equivariant spectral section

In this subsection, we extend the spectral section of Melrose-Piazza [43, 44] and the main properties of them to the equivariant case.

Definition 3.1. (Compare with [20, Definition 1.6]) Let $\mathcal{F} \in \mathrm{F}_G^*(B)$ and at least one component of the fiber has nonzero dimension. An **equivariant B-family** on $\mathcal{F}$ is a smooth family of self-adjoint pseudodifferential operators $D = \{D_b\}_{b\in B}$ on the fibers of $\mathcal{F}$, which commutes with the G-action and is first order on nonzero dimensional fibers, such that

(a) it preserves the $\mathbb{Z}_2$-grading of E when the fiber is odd dimensional;

(b) it anti-commutes with the $\mathbb{Z}_2$-grading of $\mathcal{S}_Z\widehat{\otimes}E$ when the fiber is even dimensional.

If the dimension of the fiber is zero, an equivariant B-family is a self-adjoint endomorphism of E which commutes with the G-action and anti-commutes with the $\mathbb{Z}_2$-grading of E.

If the principal symbol of D_b is the same as that of the fiberwise Dirac operator $D(\mathcal{F})|_{Z_b}$ for any $b \in B$, we call this equivariant B-family D a B-family of equivariant Dirac type operator. In this case, we have $\mathrm{Ind}(D) = \mathrm{Ind}(D(\mathcal{F})) \in K_G^*(B)$. Recall that if the fiber is a point, the fiberwise Dirac operator is zero.

Definition 3.2. (Compare with [43, Definition 1] and [44, Definition 1]) An equivariant Melrose-Piazza spectral section of an equivariant B-family $D = \{D_b\}_{b\in B}$ is a continuous family of self-adjoint pseudodifferential projections P_b on the L^2-completion of the domain of D_b, which commutes with the G-action, such that

(a) for some smooth function $f : B \to \mathbb{R}$ (depending on P) and every $b \in B$,

$$D_b u = \lambda u \Longrightarrow \begin{cases} P_b u = u, & \text{if } \lambda > f(b), \\ P_b u = 0, & \text{if } \lambda < -f(b); \end{cases} \tag{3.1}$$

(b) when the fiber is odd dimensional, P commutes with the $\mathbb{Z}_2$-grading of E;

(c) when the fiber is even dimensional,

$$\tau^{\mathcal{S}_Z\widehat{\otimes}E} \circ P + P \circ \tau^{\mathcal{S}_Z\widehat{\otimes}E} = \tau^{\mathcal{S}_Z\widehat{\otimes}E}, \tag{3.2}$$

where $\tau^{\mathcal{S}_Z\widehat{\otimes}E}$ is the $\mathbb{Z}_2$-grading of $\mathcal{S}_Z\widehat{\otimes}E$.

The following proposition is the equivariant extension of the results in [43, 44]. Remark that in our setting the dimension of the fiber might be zero. In the proof of this proposition, we will use the equivariant version of the Fredholm theory for fiberwise elliptic operators. For the references, see [3] and [26, Appendix A.5]. We will also show some details in Appendix B.

Proposition 3.3. *Let $\mathcal{F} \in \mathrm{F}_G^*(B)$ and D be an equivariant B-family on $\mathcal{F}$.*

(i) (Compare with [43, Proposition 1] and [44, Proposition 2]) If there exists an equivariant spectral section of D on $\mathcal{F} \in \mathrm{F}_G^0(B)$ (resp. $\mathrm{F}_G^1(B)$), then $\mathrm{Ind}(D) = 0 \in K_G^0(B)$ *(resp. $K_G^1(B)$). Conversely, if $\mathcal{F} \in \mathrm{F}_G^0(B)$ (resp. $\mathrm{F}_G^1(B)$),* $\mathrm{Ind}(D) = 0 \in K_G^0(B)$ *(resp. $K_G^1(B)$) and at least one component of the fibers has the nonzero dimension, there exists an equivariant spectral section of D.*

(ii) (Compare with [43, Proposition 2]) For $\mathcal{F} \in \mathrm{F}_G^1(B)$, given equivariant spectral sections P, Q of D, there exists an equivariant spectral section R of D such that $PR = R$ and $QR = R$. We say that R majors P, Q.

(iii) (Compare with [43, Lemma 8] and [44, Lemma 1]) If there exists an equivariant spectral section P of D, then there exists a family of self-adjoint equivariant smoothing operators A_P (when the dimension of the fibers are zero, it descends to a self-adjoint equivariant endmorphism of the complex vector bundle) with range in a finite sum of eigenspaces of D such that $D + A_P$ is an ***invertible*** *equivariant B-family and P is the Atiyah-Patodi-Singer (APS) projection onto the eigenspaces of the positive spectrum of $D + A_P$.*

Proof. Case 1: Let $\mathcal{F} \in \mathrm{F}^1_G(B)$. We use the notation of Appendix B in this part of the proof.

We write $\mathcal{F} = (W, o, E)$, $E = E_+ \oplus E_-$ and omit other data for simplicity. Set $\mathcal{F}_+ = (W, o, E_+ \oplus \{0\})$ and $\mathcal{F}_- = (W, o, \{0\} \oplus E_-)$. Then $\mathrm{Ind}(D(\mathcal{F})) = \mathrm{Ind}(D(\mathcal{F}_+ + \mathcal{F}_-)) = \mathrm{Ind}(D(\mathcal{F}_+)) + \mathrm{Ind}(D(\mathcal{F}_-))$. Set $\mathcal{F}_-^{\mathrm{re}} := (W, -o, E_- \oplus \{0\})$, where $-o$ is the reversion of the orientation o and E_- is equipped with the positive $\mathbb{Z}_2$-grading. Then there is a natural bijective map $\mathcal{F}_- \to \mathcal{F}_-^{\mathrm{re}}$ by identification. Note that $\mathrm{Ind}(D(\mathcal{F}_-)) = \mathrm{Ind}(D(\mathcal{F}_-^{\mathrm{re}}))$. We have $\mathrm{Ind}(D(\mathcal{F})) = \mathrm{Ind}(D(\mathcal{F}_+ + \mathcal{F}_-^{\mathrm{re}}))$. So using this bijective map, we only need to prove our proposition in odd case when $E_- = 0$ in $\mathcal{F}$.

(i) Let $T = D/(1 + D^2)^{1/2}$. Then T is bounded, G-equivariant and $\mathrm{Ind}(T) = \mathrm{Ind}(D) \in K^1_G(B)$. As in Appendix B, $\sqrt{-1}T$ can be extended to an equivariant map from B to $\mathrm{Fred}^1(L^2(G) \otimes H)$, where H is a separable Hilbert space. Assume that there exists an equivariant spectral section P of D. It could be extended on $L^2(G) \otimes C(\mathbb{R}) \otimes H$ in the same way as T, which is also denoted by P. From the definition of the equivariant spectral section, $PT(1 - P)$ and $(1 - P)TP$ are G-equivariant self-adjoint finite rank operators. Let $K = (1+D^2)^{-1/2}$ on $\mathcal{E}$. Then K is a G-equivariant self-adjoint compact operator on $L^2(G)\otimes C(\mathbb{R})\otimes H$ by taking zero on the complement of $\mathcal{E}$. Thus from the G-homotopy invariance, the equivariant family index of T in $K^1_G(B)$ is the same as that of $P(T + rK)P + (1 - P)(T - rK)(1 - P)$ for $r > 0$. When r is large enough, for any $b \in B$, $P_b(T_b + rK_b)P_b$ is positive and $(1 - P_b)(T_b - rK_b)(1 - P_b)$ is negative. Therefore we have $\mathrm{Ind}(T) = 0 \in K^1_G(B)$.

If $\mathrm{Ind}(D) = \mathrm{Ind}(T) = 0 \in K^1_G(B)$, we modify the construction of the spectral section in the proof of [43, Proposition 1]. All equivariant operators here are regarded as families of equivariant operators acting on $L^2(G) \otimes H$. Since $\mathrm{Ind}(T) = 0 \in K^1_G(B)$, from (B.1), $\sqrt{-1}T$ is G-homotopic to an invertible element in $\mathrm{Fred}^1(L^2(G) \otimes H)$ through $\sqrt{-1}T_t \in \mathrm{Fred}^1(L^2(G) \otimes H)$, $t \in [0, 1]$. As in [43], all operators in these families have discrete spectrum in some fixed open interval $(-\varepsilon, \varepsilon)$, $0 < \varepsilon < 1$. Choose $\chi \in \mathcal{C}^\infty(\mathbb{R})$ with $\chi(\lambda) = 0$ if $\lambda < 0$ and $\chi(\lambda) = 1$ if $\lambda > \varepsilon/2$. Set $J = \chi(T)$. Then J is G-equivariant and smooth on $b \in B$. From the G-homotopy above, we could construct a smooth family of G-equivariant projections P' on $L^2(G) \otimes H$ in the same way as in the proof of [43, Proposition 1] such that $J - P'$ has finite rank and the range of $J - P'$ lies in $\mathcal{E}$. By taking spectral cuts as in the proof of [33, Theorem 3], we could obtain an equivariant projection P which differs from J by an equivariant operator whose range lies in the span of a finite number of eigenfunctions of T on $\mathcal{E}$ for each $b \in B$ (see also the proof of [54, Proposition 3.7]). So $P|_{\mathcal{E}}$ is an equivariant spectral section.

(ii) We extend the equivariant spectral section P on the equivariant trivial Hilbert bundle as before and consider the family of operators PTP on the range of P. These are equivariant self-adjoint operators and from (3.1), there exists $N > 0$, such that all but the first N eigenfunctions of PTP are eigenfunctions of D. Since B is compact, we could take $0 < a_1 < 1$ such that the first N eigenvalues of PTP are all less than a_1 for any $b \in B$. Take $a_2 \in (a_1, 1)$ and choose $\chi_1 \in \mathcal{C}^\infty(\mathbb{R})$ with

$\chi_1(\lambda) = 0$ if $\lambda \leqslant a_1$ and $\chi_1(\lambda) = 1$ if $\lambda \geqslant a_2$. Then for M large enough, the range of $P - \chi_1(T)$ is an equivariant subbundle of the range of P (cf. [6, Lemma 9.9]) such that it contains the first N eigenfunctions and is contained in the span of the first M eigenfunctions of PTP. Let R be the orthogonal projection on the complement of this subbundle in $\mathcal{E}$. Then R is an equivariant spectral section such that $PR = R$. If the integer N is chosen large enough, then the projection R will have range contained in the intersection of the ranges of any two given equivariant spectral sections P and Q. So $QR = R$.

(iii) Let $\mathcal{P}_{\lambda\in[a_1,a_2],b}(D_b)$ be the span of the eigenfunctions corresponding to the eigenvalues $\lambda \in [a_1, a_2]$ of D_b. Since B is compact, we can choose $s > 0$, such that P is an equivariant spectral section for $f(b) \equiv s$. By the proof of (ii), we can choose equivariant spectral sections R', R'', such that for any $b \in B$, $R'_b = 0$ on $\mathcal{P}_{\lambda\leqslant s,b}(D_b)$ and $R''_b = I$ on $\mathcal{P}_{\lambda\geqslant -s,b}(D_b)$. Then the operator

$$\widetilde{D} = R'DR' + sPR''(I - R') + (I - R'')D(I - R'') - s(I - P)R''(I - R') \tag{3.3}$$

is an invertible equivariant B-family (cf. [43, (8.3)]). Then $A_P = \widetilde{D} - D$ satisfies all conditions.

Case 2: Let $\mathcal{F} \in \mathrm{F}^0_G(B)$ and at least one component of the fibers has nonzero dimension.

Let $D_\pm := D|_{(\mathcal{S}_Z\widehat{\otimes}E)_\pm}$. Let S be a first order positive equivariant elliptic pseudodifferential operator. Then in the sense of (B.1), D is G-homotopic to $\begin{pmatrix} S & D_- \\ D_+ & -S \end{pmatrix}$, which is invertible. Thus the equivariant K^1-index of the whole self-adjoint family D vanishes. By the same process in the proof of (i) in the odd case, there exists an equivariant spectral section P' in the odd sense, which means that it is an equivariant spectral section without the condition (3.2).

(iii) By the proof of (ii) for the odd case, we could choose P', which is an equivariant spectral section in the odd sense, such that $P'DP'$ is positive on the range of P. We simply denote by $\tau = \tau^{\mathcal{S}_Z\widehat{\otimes}E}$. Then the operator

$$A_P = P - P' - \tau(P - P')\tau + P'DP' + \tau P'\tau D\tau P'\tau - D \tag{3.4}$$

satisfies all conditions (cf. [44, (2.11), (2.12)]).

(i) Assume that $\mathrm{Ind}(D) = 0 \in K^0_G(B)$. As in the proof of (ii) for the odd case, for $r > 0$ fixed, we can choose an equivariant spectral section P' in the odd sense such that $P' = 0$ on $\mathcal{P}_{\lambda\leqslant r}(D)$. From Definition 3.1 (b), we have $\tau P'\tau = 0$ on $\mathcal{P}_{\lambda\geqslant -r}(D)$. Let $V = \ker(P' + \tau P'\tau)$. Then V is a finite dimensional equivariant complex vector bundle over B. We split the complex vector bundle by $V = V_+ \oplus V_-$ with respect to τ. Then $\mathrm{Ind}(D) = [V_+] - [V_-] \in K^0_G(B)$. The assumption $\mathrm{Ind}(D) = 0 \in K^0(B)$ implies that there exists a complex vector bundle U such that $V_+ \oplus U \simeq V_- \oplus U$ as complex vector bundles.

We choose another equivariant spectral section P'' in the odd sense such that the range of $P' - P''$ is an equivariant complex vector bundle whose rank is large enough. Let $V' = \ker(P'' + \tau P''\tau)$ and $V' = V'_+ \oplus V'_-$ with respect to τ. Let $W_\pm$ be the complex vector bundles such that $V'_\pm = V_\pm \oplus W_\pm$. Then D_+ induces an isomorphism between W_+ and W_-. Since the rank of $W_\pm$ is large enough, there exist subbundles $U_+ \subset W_+$ and $U_- \subset W_-$ such that $U_+ \simeq U_- \simeq U$ as complex vector bundles and $D_+(U_+) = U_-$. So $V'_+ \simeq V'_-$ as complex vector bundles. Since

$\operatorname{Ind}(D) = 0 \in K_G^0(B)$, this isomorphism is G-equivariant. Let

$$P_V = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix} \tag{3.5}$$

on $V' = V'_+ \oplus V'_-$. Then $P = P'' + P_V$ is an equivariant spectral section.

The other direction follows from (iii) easily.

Case 3: The dimensions of all fibers are zero.

In this case, for any self-adjoint projection $P \in \operatorname{End}(E)$ commuting with the G-action, (3.1) holds. Thus the only restriction for P as an equivariant spectral section is (3.2). If there exists an equivariant spectral section P, we take $A_P = P - \tau P \tau - D$. Thus $D + A_P = 2P - \operatorname{Id}$ is invertible and P is the projection onto the eigenspaces of the positive eigenvalues of $D + A_P$. Thus $\operatorname{Ind}(D) = 0 \in K_G^0(B)$.

The proof of Proposition 3.3 is completed. □

Remark 3.4. In zero dimensional case, we could also construct an equivariant spectral section of $D(\mathcal{F} + \mathcal{F}^{\mathrm{op}})$ as in (3.5).

Definition 3.5. Let D be an equivariant B-family on $\mathcal{F}$. A perturbation operator with respect to D is a family of bounded pseudodifferential operators A such that $D + A$ is an **invertible** equivariant B-family on $\mathcal{F}$.

Note that if there exists an equivariant spectral section of D, the smoothing operator associated with it is a perturbation operator.

Remark that the tamings in [14, 16, 18] are perturbation operators when the manifolds there are smooth, compact and without boundary.

3.2 Well-defined property for the push-forward map

In this subsection, we show that the push-forward map defined in Theorem 2.9 is well-defined. We use the notation in Section 2.4.

Lemma 3.6. *If* $\operatorname{Ind}(D(\mathcal{F}_X)) = 0 \in K_G^j(V)$*, then* $\operatorname{Ind}(D(\mathcal{F}_Z)) = 0 \in K_G^{N(j)}(B)$, $j = 0, 1$.

Proof. We only need to prove this lemma when the dimensions of the fibers are nonzero. Let

$$g_T^{TZ} = \pi_X^* g^{TY} \oplus \frac{1}{T^2} g^{TX}. \tag{3.6}$$

We denote by $C_T(TZ)$ the Clifford algebra bundle of TZ with respect to g_T^{TZ}. If $U \in TV$, let $U^H \in T_{\pi_X}^H W$ be the horizontal lift of U, such that $\pi_{X,*}(U^H) = U$. Let $\{e_i\}$, $\{f_p\}$ be local orthonormal frames of (TX, g^{TX}), (TY, g^{TY}). Then $\{f_{p,1}^H\} \cup \{Te_i\}$ is a local orthonormal frame of (TZ, g_T^{TZ}). We define a Clifford algebra isomorphism

$$\mathcal{G}_T : C_T(TZ) \to C(TZ) \tag{3.7}$$

by

$$\mathcal{G}_T(c(f_{p,1}^H)) = c(f_{p,1}^H), \quad \mathcal{G}_T(c_T(Te_i)) = c(e_i). \tag{3.8}$$

Under this isomorphism, we can consider $((\pi_X^* \mathcal{S}_Y \widehat{\otimes} \mathcal{S}_X) \widehat{\otimes} E, h^{\pi_X^* \mathcal{S}_Y \widehat{\otimes} \mathcal{S}_X} \otimes h^E)$ as a self-adjoint Hermitian equivariant Clifford module of $C_T(TZ)$. So

$$\mathcal{F}_{Z,T} = (W, L_Z, E, o_Z, T_{\pi_Z}^H W, g_T^{TZ}, h^{L_Z}, \nabla^{L_Z}, h^E, \nabla^E) \tag{3.9}$$

is an equivariant geometric family over B and $\mathcal{F}_{Z,1} = \mathcal{F}_Z$ in (2.37).

If $\mathrm{Ind}(D(\mathcal{F}_X)) = 0 \in K_G^*(V)$, from Proposition 3.3 (i), there exists a perturbation operator A_X such that $\ker(D(\mathcal{F}_X)+A_X) = 0$. We extend A_X to a pseudodifferential operator acting on $\mathcal{C}^\infty(W, (\pi_X^*\mathcal{S}_Y \widehat{\otimes} \mathcal{S}_X)\widehat{\otimes} E)$ the same way as the extension of $c(e_i)$ in Section 2.1, denoted by $1\widehat{\otimes} A_X$. From the proof of [34, Lemma 5.3], there exists $T' \geqslant 1$, such that when $T \geqslant T'$, $\ker(D(\mathcal{F}_{Z,T}) + 1\widehat{\otimes} TA_X) = 0$. So by the homotopy invariance of the equivariant family index, for any $T \geqslant 1$, we have $\mathrm{Ind}(D(\mathcal{F}_{Z,T})) = 0$.

The proof of Lemma 3.6 is completed. □

3.3 Equivariant higher spectral flow

In [20], Dai and Zhang introduced the higher spectral flow for odd dimensional fibers. In this subsection, we extend the Dai-Zhang higher spectral flow to the equivariant case and define the equivariant higher spectral flow for even dimensional fibers inspired by [44, Proposition 4].

Note that a horizontal subbundle on W is simply a splitting of the exact sequence

$$0 \to TZ \to TW \to \pi^*TB \to 0. \tag{3.10}$$

As the space of the splitting map is affine, since the G-action preserves (2.10), it follows that any pair of equivariant horizontal subbundles can be connected by a smooth path of equivariant horizontal distributions.

Assume that $\mathcal{F}, \mathcal{F}' \in \mathrm{F}_G^*(B)$ have the same topological structure, i.e., they satisfy the first three conditions in Definition 2.2. Let $r \in I$, $I = [0,1]$, parametrize a smooth path of equivariant horizontal subbundles $\{T_{\pi,r}^H W\}_{r\in[0,1]}$ such that $T_{\pi,0}^H W = T_\pi^H W$ and $T_{\pi,1}^H W = T_\pi^{'H} W$. Let g_r^{TZ}, $h_r^{L_Z}$ and h_r^E be the G-invariant metrics on TZ, L_Z and E, depending smoothly on $r \in I$, which coincide with g^{TZ}, h^{L_Z} and h^E at $r = 0$ and with $g^{'TZ}$, $h^{'L_Z}$ and $h^{'E}$ at $r = 1$. By the same reason, we can choose G-invariant Hermitian connection $\nabla_r^{L_Z}$ and ∇_r^E on L_Z and E, such that $\nabla_0^E = \nabla^E$, $\nabla_1^E = \nabla'^E$, $\nabla_0^{L_Z} = \nabla^{L_Z}$, $\nabla_1^{L_Z} = \nabla'^{L_Z}$.

Let $\widetilde{B} = B \times I$. We consider the bundle $\widetilde{\pi} : \widetilde{W} := W \times I \to \widetilde{B}$ together with the natural projection $\mathrm{Pr} : \widetilde{W} \to W$. Then the fiberwise G-action can be naturally extended to $\widetilde{\pi} : \widetilde{W} \to \widetilde{B}$ such that G acts as identity on I. Thus $T_{\widetilde{\pi}}^H \widetilde{W}_{(r,\cdot)} = \mathbb{R} \times T_{\pi,r}^H W$ defines an equivariant horizontal subbundle of $T\widetilde{W}$, and $T\widetilde{Z} := \mathrm{Pr}^*TZ$, $\widetilde{L_Z} := \mathrm{Pr}^*L_Z$ and $\widetilde{E} := \mathrm{Pr}^*E$ are naturally equipped with G-invariant metrics $g^{T\widetilde{Z}}$, $h^{\widetilde{L_Z}}$, $h^{\widetilde{E}}$ and G-invariant Hermitian connections $\nabla^{\widetilde{L_Z}}$, $\nabla^{\widetilde{E}}$. Let $\tilde{o} = \mathrm{Pr}^*o$. Then we obtain equivariant geometric families

$$\mathcal{F}_r = (W, L_Z, E, o, T_{\pi,r}^H W, g_r^{TZ}, h_r^{L_Z}, \nabla_r^{L_Z}, h_r^E, \nabla_r^E) \tag{3.11}$$

over B and

$$\widetilde{\mathcal{F}} = (\widetilde{W}, \widetilde{L_Z}, \widetilde{E}, \tilde{o}, T_{\widetilde{\pi}}^H \widetilde{W}, g^{T\widetilde{Z}}, h^{\widetilde{L_Z}}, \nabla^{\widetilde{L_Z}}, h^{\widetilde{E}}, \nabla^{\widetilde{E}}) \tag{3.12}$$

over $\widetilde{B}$ such that $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_1 = \mathcal{F}'$.

If $\mathcal{F} \in \mathrm{F}_G^1(B)$ and R, P are two equivariant spectral sections of an equivariant B-family D such that $PR = R$, then the cokernel of $P_bR_b : \mathrm{Im}(R_b) \to \mathrm{Im}(P_b)$ for $b \in B$ forms an equivariant complex vector bundle over B, denoted by $[P - R]$. Hence for any two equivariant spectral sections P, Q, the difference element $[P-Q]$ can be defined as an element in $K_G^0(B)$ as follows:

$$[P - Q] := [P - R] - [Q - R] \in K_G^0(B), \tag{3.13}$$

where R is an equivariant spectral section which majors P, Q as in Proposition 3.3 (ii) . From (3.13), we can obtain that if P_1, P_2, P_3 are equivariant spectral sections of D, then

$$[P_3 - P_1] = [P_3 - P_2] + [P_2 - P_1] \in K_G^0(B). \tag{3.14}$$

Thus the class in (3.13) is independent of the choice of R.

Let $\mathcal{F}, \mathcal{F}' \in \mathrm{F}_G^1(B)$ which have the same topological structure. Let $\mathcal{F}_r$ and $\widetilde{\mathcal{F}}$ are equivariant geometric families in (3.11) and (3.12). Now we consider a continuous family of operators D_r on $\mathcal{F}_r$ for $r \in I$ such that D_r is an equivariant B-family on $\mathcal{F}_r$. Assume that $\mathrm{Ind}(D_0) = 0 \in K_G^1(B)$. Then the homotopy invariance of the equivariant family index implies that the equivariant indice of D_r vanish. Let Q_0 and Q_1 be equivariant spectral sections of D_0 and D_1 respectively. If we consider the total family $\widetilde{D} = \{D_r\}$ parametrized by $B \times I$, then there exists a total equivariant spectral section $\widetilde{P}$. Let P_r be the restriction of $\widetilde{P}$ over $B \times \{r\}$. Thus we have the natural equivariant extension of the higher spectral flow in [20, Definition 1.5].

Definition 3.7. The equivariant Dai-Zhang higher spectral flow $\mathrm{sf}_G\{(D_0, Q_0), (D_1, Q_1)\}$ between the pairs (D_0, Q_0), (D_1, Q_1) is an element in $K_G^0(B)$ defined by

$$\mathrm{sf}_G\{(D_0, Q_0), (D_1, Q_1)\} = [Q_1 - P_1] - [Q_0 - P_0] \in K_G^0(B). \tag{3.15}$$

From (3.14), we know that this definition is independent of the choice of the total equivariant spectral section $\widetilde{P}$.

In the following, we define the equivariant higher spectral flow for the even case.

Let $\mathcal{F} \in \mathrm{F}_G^0(B)$. Let D be an equivariant B-family on $\mathcal{F}$. **We assume that there exists an equivariant spectral section P with respect to D.** Let A_P be the family of self-adjoint equivariant smoothing operators associated with P by Proposition 3.3 (iii).

Now we use the notation in Example 2.5 d). Let $p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L$ be the odd equivariant geometric family in Example 2.5 d) with fibers $Z \times S^1$. Let τ be the $\mathbb{Z}_2$-grading of the $\mathcal{S}_Z \widehat{\otimes} E$ in $\mathcal{F}$. We consider the vector bundle part in $p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L$ as an ungraded one. Then from Definition 3.1,

$$D_P = (D + A_P) \otimes 1 + \tau \otimes D(\mathcal{F}^L) \tag{3.16}$$

is an equivariant $B \times S^1$-family on the odd geometric family $p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L$ and commutes with the group action.

Since D and A_P anti-commute with τ,

$$D_P^2 = ((D + A_P) \otimes 1 + \tau \otimes D(\mathcal{F}^L))^2 = (D + A_P)^2 \otimes 1 + 1 \otimes D(\mathcal{F}^L)^2 > 0. \tag{3.17}$$

It implies that D_P is invertible. Thus the APS projection P' is an equivariant spectral section of D_P. Similarly, let Q be another equivariant spectral section of D, we can construct the equivariant spectral section Q' of D_Q as above. Since $p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L \in \mathrm{F}_G^1(B)$, from Definition 3.7, we could define $\mathrm{sf}_G\{(D_P, P'), (D_Q, Q')\} \in K_G^0(B \times S^1)$.

Now we consider Example 2.5 c) more explicitly. It is easy to calculate that for $\theta \in [0, 1)$ fixed, the eigenvalues of $D(\mathcal{F}^L)$ are $\lambda_k(\theta) = 2\pi k + 2\pi(\theta - 1/2)$, $k \in \mathbb{Z}$. So for $\theta \in [0, 1)$, $\theta \neq 1/2$, we have $D(\mathcal{F}^L)^2 > 0$. Thus as in (3.17), for any $s \in [0, 1]$, $\theta \neq 1/2$, restricted on $B \times \{\theta\}$, $(1 - s)D_P + sD_Q$ is invertible. From Definition 3.7, it means that for $\theta \neq 1/2$, $\mathrm{sf}_G\{(D_P, P'), (D_Q, Q')\}|_{B\times\{\theta\}} = 0 \in K_G^0(B \times \{\theta\})$.

From (2.19), there exists an element in $K^1_G(B)$, which we denote by $[Q-P]$, such that

$$j([Q-P]) = \mathrm{sf}_G\{(D_P, P'), (D_Q, Q')\} \in K^0_G(B\times S^1). \tag{3.18}$$

The idea for this construction comes from [44, Proposition 4]. We note that when the group G is trivial, this definition is equivalent to that there.

Similarly, if P_1, P_2, P_3 are equivariant spectral sections of D, then

$$[P_3 - P_1] = [P_3 - P_2] + [P_2 - P_1] \in K^1_G(B). \tag{3.19}$$

Now we extend the difference $[Q-P]$ to the equivariant higher spectral flow. Let $\mathcal{F}, \mathcal{F}' \in \mathrm{F}^0_G(B)$, which have the same topological structure, and D_0, D_1 be two equivariant B-families on $\mathcal{F}$, $\mathcal{F}'$ respectively. For $i=0,1$, let Q_i be an equivariant spectral section of D_i with corresponding smoothing operators A_{Q_i}. Let $D(r)$, $r\in[0,1]$ be a continuous curve of equivariant B-families on $\mathcal{F}_r$ such that $D(i) = D_i + A_{Q_i}$, $i=0,1$. Let

$$D_{i,Q_i} = (D_i + A_{Q_i}) \otimes 1 + \tau \otimes D(\mathcal{F}^L). \tag{3.20}$$

By (3.17), they are invertible. Let Q'_i be their APS projections. Let $\widetilde{D} = \{D(r)\otimes 1 + \tau\otimes D(\mathcal{F}^L)\}$ parametrized by $B\times S^1\times I$. By (3.17), D_{0,Q_0} is invertible. Thus $\mathrm{Ind}(D_{0,Q_0}) = 0 \in K^1_G(B\times S^1)$. So $\mathrm{Ind}(\widetilde{D}) = 0 \in K^1_G(B\times S^1\times I)$. Let $\widetilde{P} = \{P(r)\}_{r\in[0,1]}$ be an equivariant spectral section with respect to $\widetilde{D}$ such that $P(i)$ majors Q'_i for $i=0,1$. Then from Definition 3.7,

$$\mathrm{sf}_G\{(D_{0,Q_0}, Q'_0), (D_{1,Q_1}, Q'_1)\} = [Q'_1 - P(1)] - [Q'_0 - P(0)] \in K^0_G(B\times S^1). \tag{3.21}$$

Furthermore, we could obtain that this equivariant higher spectral flow lies in the image of j in (2.19). In fact, when restricted on $B\times\{\theta\}\times I$ for $\theta\neq 1/2$, as in (3.17), $\widetilde{D}|_{B\times\{\theta\}\times I}$ is invertible. For $\theta\neq 1/2$, let $\{P'(r)_\theta\}_{r\in[0,1]}$ be the APS projection of $\widetilde{D}|_{B\times\{\theta\}\times I}$. Then $P'(0)_\theta = Q'_0|_{B\times\{\theta\}}$ and $P'(1)_\theta = Q'_1|_{B\times\{\theta\}}$. Since $P'(r)_\theta$ and $P(r)|_{B\times\{\theta\}\times I}$ are two equivariant spectral sections of $\widetilde{D}|_{B\times\{\theta\}\times I}$ and $P(i)|_{B\times\{\theta\}}$ majors $Q'_i|_{B\times\{\theta\}} = P'(i)_\theta$ for $i=0,1$, we see that $[P'(r)_\theta - P(r)|_{B\times\{\theta\}\times I}]$ forms an equivariant complex vector bundle over $B\times\{\theta\}\times I$. Thus we have $([Q'_1 - P(1)] - [Q'_0 - P(0)])|_{B\times\{\theta\}} = 0 \in K^0_G(B\times\{\theta\})$. It implies that $\mathrm{sf}_G\{(D_{0,Q_0}, Q'_0), (D_{1,Q_1}, Q'_1)\} \in \mathrm{Im}(j)$.

Definition 3.8. If $\mathcal{F}, \mathcal{F}' \in \mathrm{F}^0_G(B)$, the equivariant higher spectral flow $\mathrm{sf}_G\{(D_0,Q_0),(D_1,Q_1)\}$ between the pairs (D_0,Q_0), (D_1,Q_1) is an element in $K^1_G(B)$ defined by

$$j\big(\mathrm{sf}_G\{(D_0,Q_0),(D_1,Q_1)\}\big) = \mathrm{sf}_G\{(D_{0,Q_0}, Q'_0), (D_{1,Q_1}, Q'_1)\}. \tag{3.22}$$

Note that when $\mathcal{F} = \mathcal{F}'$, $D_0 = D_1 = D$, the equivariant higher spectral flow $\mathrm{sf}_G\{(D,Q_0),(D,Q_1)\} = [Q_1 - Q_0]$.

The following proposition says that any element of equivariant K-group could be generated by equivariant higher spectral flows. Our proof is constructive.

Proposition 3.9. *(i) For any $x\in K^0_G(B)$, there exist $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{F}^1_G(B)$ and equivariant spectral sections P_i, Q_i with respect to $D(\mathcal{F}_i)$ for $i=1,2$, such that $x = [P_1 - Q_1] - [P_2 - Q_2]$.*

(ii) For any $x\in K^1_G(B)$, there exist $\mathcal{F}\in \mathrm{F}^0_G(B)$ and equivariant spectral sections P, Q with respect to $D(\mathcal{F})$, such that $x = [P-Q]$.

Proof. Let (E, h^E) be a Hermitian vector bundle and ∇^E be a Hermitian connection on (E,h^E). Let $\pi : B\times S^1 \to B$ be the projection onto the first part. Let

$\mathcal{F}=(B\times S^1,\pi^*E,o,T^H(B\times S^1),g^{TS^1},\pi^*h^E,\pi^*\nabla^E)\in \mathrm{F}^1_G(B)$, where o, g^{TS^1} are the canonical orientation and metric on S^1 and $T^H(B\times S^1)=TB\times S^1$. Let ∂_t be the generator of TS^1. Then $D(\mathcal{F})=-\sqrt{-1}\partial_t\otimes \mathrm{Id}_E$. We could calculate that the eigenvalues of $D(\mathcal{F})$ are $\lambda_k=k$ for $k\in\mathbb{Z}$. We denote by $P_{\lambda\geqslant k}$ the orthogonal projection onto the union of the eigenspaces of $\lambda\geqslant k$. Then for any k, $P_{\lambda\geqslant k}$ is an equivariant spectral section of $D(\mathcal{F})$. In particular, we have $[P_{\lambda\geqslant k}-P_{\lambda\geqslant k+1}]=[E]\in K^0_G(B)$. Thus we obtain Proposition 3.9 in the even case.

For any $x\in K^1_G(B)$, from Lemma 2.4, there exists a finite dimensional complex unitary representation V of G, such that x can be represented as a G-invariant unitary element $F\in\mathcal{C}^\infty(B,\mathrm{End}(V))$. Let $\mathcal{F}_1=(B,E_+=E_-=B\times V)\in \mathrm{F}^0_G(B)$, with fiber $Z=\mathrm{pt}$ and trivial metric and connection on $E_\pm$. Let

$$A_0=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad A_1=\begin{pmatrix}0 & F(b)^*\\ F(b) & 0\end{pmatrix}$$

be endomorphisms of $V\oplus V$. Let P_i be the orthogonal projection onto the positive part of the spectrum of A_i for $i=0,1$. It is easy to calculate that for $i=0,1$, $P_i\tau+\tau P_i=\tau$. From Definition 3.2, we know that P_0 and P_1 are equivariant spectral sections with respect to $D(\mathcal{F}_1)=0$ on $\mathcal{F}_1$. Let $D_i=A_i\otimes 1+\tau\otimes D(\mathcal{F}^L)$ on $p_1^*\mathcal{F}_1\times_{B\times S^1}p_2^*\mathcal{F}^L$ and P_i' be the APS projections of D_i. Let $D_s=(1-s)D_0+sD_1$ for $s\in[0,1]$. We claim that

$$\mathrm{sf}_G\{(D_0,P_0'),(D_1,P_1')\}=[W]-[U]\in K^0_G(B\times S^1),\tag{3.23}$$

where W and U are bundles constructed above Lemma 2.4. Then from (3.18) and (3.23), we obtain Proposition 3.9 in the odd case.

We prove the claim (3.23) constructively. Let $\lambda_{b,i}$ be the eigenvalues of $F(b)$ on V with unitary eigenvectors $v_{b,i}$. Then $\overline{\lambda}_{b,i}$ are the eigenvalues of $F^*(b)$ on V with the same eigenvectors. Let $v^\pm_{b,i}$ be the corresponding vectors in $E_{\pm,b}$. Let v_k be the eigenvector of $\lambda_k(\theta)$ with respect to $D(\mathcal{F}^L)$ (see Appendix B). From (3.17), it is easy to calculate that the nonnegative eigenvalues of D_s are

$$\lambda_{s,b,i,k}(\theta)=\sqrt{\lambda_k(\theta)^2+(1-2s)^2+s(1-s)(\lambda_{b,i}+\overline{\lambda}_{b,i}+2)}.\tag{3.24}$$

Since F is unitary, $|\lambda_{b,i}|=1$. So $\lambda_{s,b,i,k}(\theta)=0$ if and only if $k=0,\theta=\frac{1}{2},s=\frac{1}{2}$ and $\lambda_{b,i}=-1$. From (3.16), we calculate that the eigenfunctions of $\lambda_{s,b,i,1}(\theta)$ with respect to D_s are

$$\begin{aligned}u^{(1)}_{s,b,i}(\theta)&=((s\overline{\lambda}_{b,i}+1-s)v^+_{b,i}+(\lambda_{s,b,i,1}(\theta)-\lambda_1(\theta))v^-_{b,i})\otimes v_1,\\ u^{(2)}_{s,b,i}(\theta)&=((\lambda_{s,b,i,-1}(\theta)+\lambda_{-1}(\theta))v^+_{b,i}+(s\lambda_{b,i}+1-s)v^-_{b,i})\otimes v_{-1}.\end{aligned}\tag{3.25}$$

Let

$$\begin{aligned}u^{(3)}_{s,b,i}(\theta)&=((s\overline{\lambda}_{b,i}+1-s)v^+_{b,i}+(\lambda_{s,b,i,0}(\theta)-\lambda_0(\theta))v^-_{b,i})\otimes v_0,\ 0\leqslant\theta\leqslant 1/2,\\ u^{(4)}_{s,b,i}(\theta)&=((\lambda_{s,b,i,0}(\theta)+\lambda_0(\theta))v^+_{b,i}+(s\lambda_{b,i}+1-s)v^-_{b,i})\otimes v_0,\ 1/2\leqslant\theta\leqslant 1.\end{aligned}\tag{3.26}$$

Then $u^{(3)}_{s,b,i}(\theta)$ and $u^{(4)}_{s,b,i}(\theta)$ are the eigenfunctions of $\lambda_{s,b,i,0}(\theta)$ with respect to D_s. Let

$$w^{(j)}_{s,b,i}(\theta)=u^{(j)}_{s,b,i}(\theta)/\|u^{(j)}_{s,b,i}(\theta)\|,\quad j=1,2,3,4.\tag{3.27}$$

Remark that when $\theta=1/2$, if $s=1/2$ and $\lambda_{b,i}=-1$, then $u^{(3)}_{s,b,i}(1/2)=u^{(4)}_{s,b,i}(1/2)=0$. In this case, we define

$$\begin{aligned}w^{(3)}_{s,b,i}(1/2)&=\lim_{\theta\to 1/2^-}u^{(3)}_{s,b,i}(\theta)/\|u^{(3)}_{s,b,i}(\theta)\|,\\ w^{(4)}_{s,b,i}(1/2)&=\lim_{\theta\to 1/2^+}u^{(4)}_{s,b,i}(\theta)/\|u^{(4)}_{s,b,i}(\theta)\|.\end{aligned}\tag{3.28}$$

Choose $\chi(\theta) \in \mathcal{C}^\infty([0,1/2])$ with $\chi(\theta) = 1/2$ near $\theta = 0$ and $\chi(\theta) = 0$ near $\theta = 1/2$. Let

$$\begin{aligned} w^{(5)}_{s,b,i}(\theta) &= \chi(\theta) w^{(1)}_{s,b,i}(\theta) + (1-\chi(\theta)) w^{(3)}_{s,b,i}(\theta), && 0 \leqslant \theta \leqslant 1/2, \\ w^{(6)}_{s,b,i}(\theta) &= \chi(1-\theta) w^{(2)}_{s,b,i}(\theta) + (1-\chi(1-\theta)) w^{(4)}_{s,b,i}(\theta), && 1/2 \leqslant \theta \leqslant 1. \end{aligned} \tag{3.29}$$

Since $v_1(\theta = 0) = v_0(\theta = 1)$, $v_0(\theta = 0) = v_{-1}(\theta = 1)$, $\lambda_0(0) = \lambda_{-1}(1) = -\pi$ and $\lambda_0(1) = \lambda_1(0) = \pi$, from (3.25)-(3.27), we have $w^{(1)}_{s,b,i}(0) = w^{(3)}_{s,b,i}(1)$ and $w^{(2)}_{s,b,i}(1) = w^{(4)}_{s,b,i}(0)$. By (3.29), we have

$$w^{(5)}_{s,b,i}(0) = w^{(6)}_{s,b,i}(1). \tag{3.30}$$

So $\bigoplus_i \mathbb{C}\{w^{(5)}_{s,b,i}(\theta), 0 \leqslant \theta < 1/2\}$ and $\bigoplus_i \mathbb{C}\{w^{(6)}_{s,b,i}(\theta), 1/2 < \theta \leqslant 1\}$ can be connected as a trivial equivariant complex vector bundle over $B \times (S^1_\theta \backslash \{1/2\}) \times [0,1]_s$. Then we could glue $w^{(5)}_{s,b,i}(1/2)$ and $w^{(6)}_{s,b,i}(1/2)$ for any i to get an equivariant complex vector bundle $\widetilde{W}$ over $B \times S^1_\theta \times [0,1]_s$. Let $\widetilde{R}$ be the orthogonal projection onto the sum of $\widetilde{W}$ and the eigenspaces with non-positive eigenvalues of $\widetilde{D} = \{D_s\}$. Then $\widetilde{Q} = 1 - \widetilde{R}$ is an equivariant spectral section with respect to $\widetilde{D}$. Since $\operatorname{Ker} D_s \neq \emptyset$ only when $s = 1/2$, from (3.21), we have

$$\begin{aligned} \mathrm{sf}_G\{(D_0, P_0'), (D_1, P_1')\} &= [P_1' - \widetilde{Q}|_{s=1}] - [P_0' - \widetilde{Q}|_{s=0}] \\ &= [\widetilde{W}|_{s=1}] - [\widetilde{W}|_{s=0}]. \end{aligned} \tag{3.31}$$

For $s = 1$, from (3.26), (3.27) and (3.29), we have $w^{(5)}_{0,b,i}(1/2) = w^{(3)}_{0,b,i}(1/2) = (\bar{\lambda}_{b,i} v^+_{b,i} + v^-_{b,i})/\sqrt{2}$ and $w^{(6)}_{0,b,i}(1/2) = w^{(4)}_{0,b,i}(1/2) = (v^+_{b,i} + \lambda_{b,i} v^-_{b,i})/\sqrt{2}$. So $w^{(6)}_{0,b,i}(1/2) = \lambda_{b,i} \cdot w^{(5)}_{0,b,i}(1/2)$. From the construction before Lemma 2.4, we have $[\widetilde{W}|_{s=1}] = [W]$.

For $s = 0$, in the same way, we calculate that $w^{(6)}_{1,b,i}(1/2) = w^{(5)}_{1,b,i}(1/2) = (v^+_{b,i} + v^-_{b,i})/\sqrt{2}$. So $[\widetilde{W}|_{s=0}] = [U]$.

Therefore, we obtain the claim (3.23) from (3.31).

The proof of Proposition 3.9 is completed. □

Note that the proof of Proposition 3.9 in odd case gives a nontrivial example of the equivariant higher spectral flow for even dimensional fibers and an example of the equivariant spectral section without the spectral gap.

Remark 3.10. In non-equivariant case, there is a stronger version of Proposition 3.9 in [44, Proposition 12].

3.4 Equivariant local family index theorem

In this subsection, we use the notation in Section 1.2 to describe the equivariant local index theorem for $\mathcal{F} \in \mathrm{F}^*_G(B)$ when **the G-action on B is trivial**.

For $b \in B$, let $\mathcal{E}_b$ be the set of smooth sections over Z_b of $\mathcal{S}_Z \widehat{\otimes} E|_{Z_b}$. As in [8], we will regard $\mathcal{E}$ as an infinite dimensional vector bundle over B.

Let ∇^{TB} be the Levi-Civita connection on (B, g^{TB}). Let ${}^0\nabla^{TW}$ be the connection on $TW = T^H W \oplus TZ$ defined by

$${}^0\nabla^{TW} = \pi^* \nabla^{TB} \oplus \nabla^{TZ}. \tag{3.32}$$

Then ${}^0\nabla^{TW}$ preserves the metric g^{TW} in (2.13). Set

$$S = \nabla^{TW} - {}^0\nabla^{TW}. \tag{3.33}$$

If $V \in TB$, let $V^H \in T^H_\pi W$ be its horizontal lift in $T^H_\pi W$ so that $\pi_* V^H = V$. For any $V \in TB$, $s \in \mathcal{C}^\infty(B, \mathcal{E}) = \mathcal{C}^\infty(W, \mathcal{S}_Z \widehat{\otimes} E)$, by [10, Proposition 1.4], the connection

$$\nabla^{\mathcal{E}}_V s := \nabla^{\mathcal{S}_Z \widehat{\otimes} E}_{V^H} s - \frac{1}{2} \langle S(e_i) e_i, V^H \rangle \, s \tag{3.34}$$

preserves the L^2-product on $\mathcal{E}$.

Let $\{f_p\}$ be a local orthonormal frame of TB and $\{f^p\}$ be its dual. We denote by $\nabla^{\mathcal{E}} = f^p \wedge \nabla^{\mathcal{E}}_{f_p}$. Let T be the torsion of ${}^0\nabla^{TW}$. Then $T(f^H_p, f^H_q) \in TZ$. We denote by

$$c(T) = \frac{1}{2} \, c\left(T(f^H_p, f^H_q)\right) f^p \wedge f^q \wedge . \tag{3.35}$$

By [8, (3.18)], the (rescaled) Bismut superconnection

$$\mathbb{B}_u : \mathcal{C}^\infty(B, \Lambda(T^*B) \widehat{\otimes} \mathcal{E}) \to \mathcal{C}^\infty(B, \Lambda(T^*B) \widehat{\otimes} \mathcal{E}) \tag{3.36}$$

is defined by

$$\mathbb{B}_u = \sqrt{u} D(\mathcal{F}) + \nabla^{\mathcal{E}} - \frac{1}{4\sqrt{u}} c(T). \tag{3.37}$$

Obviously, the Bismut superconnection $\mathbb{B}_u$ commutes with the G-action. Moreover, $\mathbb{B}^2_u$ is a 2-order elliptic differential operator along the fibers Z (cf. [8, (3.4)]). Let $\exp(-\mathbb{B}^2_u)$ be the family of heat operators associated with the fiberwise elliptic operator $\mathbb{B}^2_u$. From [6, Theorem 9.50], $\exp(-\mathbb{B}^2_u)$ is a smooth family of smoothing operators.

If P is a trace class operator acting on $\Lambda(T^*B) \widehat{\otimes} \operatorname{End}(\mathcal{E})$ which takes values in $\Lambda(T^*B)$, we use the convention that if $\omega \in \Lambda(T^*B)$,

$$\operatorname{Tr}_s[\omega P] = \omega \operatorname{Tr}_s[P]. \tag{3.38}$$

We denote by $\operatorname{Tr}_s^{\mathrm{odd/even}}[P]$ the part of $\operatorname{Tr}_s[P]$ which takes values in odd or even forms. Set

$$\widetilde{\operatorname{Tr}}[P] = \begin{cases} \operatorname{Tr}_s[P], & \text{if } \dim Z \text{ is even;} \\ \operatorname{Tr}_s^{\mathrm{odd}}[P], & \text{if } \dim Z \text{ is odd.} \end{cases} \tag{3.39}$$

Recall that in this subsection we assume that G acts trivially on B. Take $g \in G$. Let W^g be the fixed point set of g on W. Then W^g is a submanifold of W and $\pi : W^g \to B$ is a fiber bundle with compact fibers Z^g. Set

$$\operatorname{ch}_g(E, \nabla^E) = \operatorname{Tr}_s\left[g \exp\left(\frac{\sqrt{-1}}{2\pi} R^E|_{W^g}\right)\right]. \tag{3.40}$$

Let $\operatorname{ch}_g(E) \in H^{even}(W^g, \mathbb{C})$ denote the cohomology class of $\operatorname{ch}_g(E, \nabla^E)$. When the fiber Z is a point, it descends to the equivariant Chern character map

$$\operatorname{ch}_g : K^0_G(B) \longrightarrow H^{even}(B, \mathbb{C}). \tag{3.41}$$

By (2.19), for $x \in K^1_G(B)$, $j(x) \in K^0_G(B \times S^1)$. The odd equivariant Chern character map

$$\operatorname{ch}_g : K^1_G(B) \longrightarrow H^{odd}(B, \mathbb{C}) \tag{3.42}$$

is defined by

$$\mathrm{ch}_g(x) := \int_{S^1} \mathrm{ch}_g(j(x)). \tag{3.43}$$

We adopt the sign notation in the integral as in (1.7). This is just the equivariant version of the odd Chern character in [28] and [53, (1.50)] (see e.g., [36, (3.10)]).

Let N be the normal bundle of W^g in W. As G is compact, there is an orthonormal decomposition of real vector bundles over W^g,

$$TZ|_{W^g} = TZ^g \oplus N. \tag{3.44}$$

Let ∇ be a Euclidean connection on (TZ, g^{TZ}) commuting with the G-action. Then its restriction on W^g preserves the decomposition (3.44). Let ∇^{TZ^g} and ∇^N be the corresponding induced connections on TZ^g and N, with curvatures R^{TZ^g} and R^N respectively. Set

$$\begin{aligned}\widehat{\mathrm{A}}_g(TZ,\nabla) := {}& \det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TZ^g}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{TZ^g}\right)}\right)\\ &\times\left(\sqrt{-1}^{\frac{1}{2}\dim N}\det{}^{1/2}\Big|_N\left(1-g\exp\left(\frac{\sqrt{-1}}{2\pi}R^N\right)\right)\right)^{-1}.\end{aligned} \tag{3.45}$$

If g acts on $L|_{W^g}$ by multiplying by $e^{\sqrt{-1}v}$, we write

$$\mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) := \exp\left(\frac{\sqrt{-1}}{4\pi}R^L|_{W^g} + \frac{\sqrt{-1}}{2}v\right). \tag{3.46}$$

We denote by

$$\mathrm{Td}_g(\nabla, \nabla^{L_Z}) := \widehat{\mathrm{A}}_g(TZ,\nabla)\,\mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}). \tag{3.47}$$

Let $\mathrm{Td}_g(TZ, L_Z) \in H^{even}(W^g, \mathbb{C})$ denote the cohomology class of $\mathrm{Td}_g(\nabla, \nabla^{L_Z})$.

For $\alpha \in \Omega^i(B)$, set

$$\psi_B(\alpha) = \begin{cases} \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{i}{2}} \cdot \alpha, & \text{if } i \text{ is even;} \\ \frac{1}{\sqrt{\pi}}\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{i-1}{2}} \cdot \alpha, & \text{if } i \text{ is odd.}\end{cases} \tag{3.48}$$

We state the equivariant family local index theorem here (cf. e.g., [8, Theorem 4.17], [11, Theorem 2.10], [34, Theorem 2.2], [35, Theorem 2.2] and [37, Theorem 1.3]). Note that from [38, Lemma 4.1], Z^g is naturally oriented.

Theorem 3.11. *For any $u > 0$ and $g \in G$, the differential form $\psi_B\widetilde{\mathrm{Tr}}[g\exp(-\mathbb{B}_u^2)] \in \Omega^*(B,\mathbb{C})$ is closed and its cohomology class represents $\mathrm{ch}_g(\mathrm{Ind}(D(\mathcal{F}))) \in H^*(B,\mathbb{C})$. As $u \to 0$, we have*

$$\lim_{u\to 0}\psi_B\widetilde{\mathrm{Tr}}[g\exp(-\mathbb{B}_u^2)] = \int_{Z^g}\mathrm{Td}_g(\nabla^{TZ}, \nabla^{L_Z})\ \mathrm{ch}_g(E, \nabla^E). \tag{3.49}$$

To simplify the notations, we set

$$\mathrm{FLI}_g(\mathcal{F}) = \int_{Z^g}\mathrm{Td}_g(\nabla^{TZ}, \nabla^{L_Z})\ \mathrm{ch}_g(E, \nabla^E) \in \Omega^*(B,\mathbb{C}). \tag{3.50}$$

So Theorem 3.11 says that for $\mathcal{F} \in \mathrm{F}_G^{0/1}(B)$,

$$[\mathrm{FLI}_g(\mathcal{F})] = \mathrm{ch}_g(\mathrm{Ind}(D(\mathcal{F}))) \in H^{even/odd}(B,\mathbb{C}). \tag{3.51}$$

When $\mathcal{F}$ is the equivariant geometric family in Example 2.5 a), $Z = \mathrm{pt}$, the equivariant family local index theorem degenerates to the equivariant Chern-Weil theory:

$$\psi_B \widetilde{\mathrm{Tr}}[g\exp(-\mathbb{B}_u^2)] = \psi_B \,\mathrm{Tr}_s[g\exp(-(\nabla^E)^2)] = \mathrm{ch}_g(E, \nabla^E). \tag{3.52}$$

In this case, $\mathrm{FLI}_g(\mathcal{F}) = \mathrm{ch}_g(E, \nabla^E) = \mathrm{ch}_g(E_+, \nabla^{E_+}) - \mathrm{ch}_g(E_-, \nabla^{E_-})$.

If $\alpha \in \Lambda(T^*(\mathbb{R}_+ \times B))$,

$$\alpha = \alpha_0 + ds \wedge \alpha_1, \quad \alpha_0, \alpha_1 \in \Lambda(T^*B). \tag{3.53}$$

Set

$$[\alpha]^{ds} = \alpha_1. \tag{3.54}$$

Let $\mathcal{F}, \mathcal{F}' \in \mathrm{F}_G^*(B)$ which have the same topological structure. By (3.51), we have $[\mathrm{FLI}_g(\mathcal{F})] = [\mathrm{FLI}_g(\mathcal{F}')] \in H^*(B, \mathbb{C})$.

We use the notation in (3.11) and (3.12). By [42, Theorem B.5.4], modulo exact forms on W^g, the equivariant Chern-Simons forms

$$\begin{aligned} &\widetilde{\mathrm{Td}}_g(\nabla^{TZ}, \nabla^{L_Z}, \nabla'^{TZ}, \nabla'^{L_Z}) := -\int_0^1 [\mathrm{Td}_g(\nabla^{T\widetilde{Z}}, \nabla^{\widetilde{L_Z}})]^{ds} ds, \\ &\widetilde{\mathrm{ch}}_g(\nabla^E, \nabla'^E) := -\int_0^1 [\mathrm{ch}_g(\widetilde{E}, \nabla^{\widetilde{E}})]^{ds} ds \end{aligned} \tag{3.55}$$

depend only on the connections in $\mathcal{F}$ and $\mathcal{F}'$. Moreover,

$$\begin{aligned} &d^{W^g}\, \widetilde{\mathrm{Td}}_g(\nabla^{TZ}, \nabla^{L_Z}, \nabla'^{TZ}, \nabla'^{L_Z}) = \mathrm{Td}_g(\nabla'^{TZ}, \nabla'^{L_Z}) - \mathrm{Td}_g(\nabla^{TZ}, \nabla^{L_Z}), \\ &d^{W^g}\, \widetilde{\mathrm{ch}}_g(\nabla^E, \nabla'^E) = \mathrm{ch}_g(E, \nabla'^E) - \mathrm{ch}_g(E, \nabla^E). \end{aligned} \tag{3.56}$$

Set

$$\begin{aligned} \widetilde{\mathrm{FLI}}_g(\mathcal{F}, \mathcal{F}') = &\int_{Z^g} \widetilde{\mathrm{Td}}_g(\nabla^{TZ}, \nabla^{L_Z}, \nabla'^{TZ}, \nabla'^{L_Z})\, \mathrm{ch}_g(E, \nabla^E) \\ &+ \int_{Z^g} \mathrm{Td}_g(\nabla'^{TZ}, \nabla'^{L_Z})\, \widetilde{\mathrm{ch}}_g(\nabla^E, \nabla'^E) \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}). \end{aligned} \tag{3.57}$$

From [6, (1.7)], for $\sigma \in \Omega^*(W^g)$, using the sign convention in (1.7), we have

$$d^B \int_{Z^g} \sigma = \int_{Z^g} d^{W^g} \sigma. \tag{3.58}$$

From (3.56) and (3.58), we have

$$d^B\, \widetilde{\mathrm{FLI}}_g(\mathcal{F}, \mathcal{F}') = \mathrm{FLI}_g(\mathcal{F}') - \mathrm{FLI}_g(\mathcal{F}). \tag{3.59}$$

3.5 Equivariant eta form

In this subsection, we also assume that G acts trivially on B. We define the equivariant Bismut-Cheeger eta form with perturbation operator in Definition 3.5.

In this subsection, **we assume that there exists a perturbation operator with respect to $D(\mathcal{F})$ on $\mathcal{F}$.** It implies that $\mathrm{Ind}(D(\mathcal{F})) = 0 \in K_G^*(B)$.

Let A be a perturbation operator with respect to $D(\mathcal{F})$. We extend A to $1\widehat{\otimes}A$ on $\mathcal{C}^\infty(B, \pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E})$ as an element of the $\mathbb{Z}_2$-graded tensor product of $\mathbb{Z}_2$-graded algebras. In this case,

$$(\alpha\widehat{\otimes}1)(1\widehat{\otimes}A) = (-1)^{\deg \alpha}(1\widehat{\otimes}A)(\alpha\widehat{\otimes}1). \tag{3.60}$$

We usually abbreviate $1\widehat{\otimes}A$ by A when there is no confusion.

Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ be a cut-off function such that

$$\chi(u) = \begin{cases} 0, & \text{if } u < 1; \\ 1, & \text{if } u > 2. \end{cases} \tag{3.61}$$

Set

$$\mathbb{B}'_u = \mathbb{B}_u + \sqrt{u}\chi(\sqrt{u})A. \tag{3.62}$$

Since $\chi(\sqrt{u}) = 0$ if $u \in (0,1)$, by (3.49) and (3.50),

$$\lim_{u\to 0} \psi_B \widetilde{\mathrm{Tr}}[g\exp(-(\mathbb{B}'_u)^2)] = \mathrm{FLI}_g(\mathcal{F}) \in \Omega^*(B,\mathbb{C}). \tag{3.63}$$

Since $\chi(\sqrt{u}) = 1$ if $u \in (2,+\infty)$, from [6, Theorem 9.19], we have

$$\lim_{u\to +\infty} \psi_B \widetilde{\mathrm{Tr}}\left[g\exp\left(-(\mathbb{B}'_u)^2\right)\right] = 0. \tag{3.64}$$

Definition 3.12. For any $g \in G$, modulo exact forms on B, the equivariant eta form with perturbation operator A is defined by

$$\tilde{\eta}_g(\mathcal{F},A) = -\int_0^\infty \left\{\psi_{\mathbb{R}\times B}\,\widetilde{\mathrm{Tr}}\left[g\exp\left(-\left(\mathbb{B}'_u + du\wedge\frac{\partial}{\partial u}\right)^2\right)\right]\right\}^{du} du \\ \in \Omega^*(B,\mathbb{C})/d\Omega^*(B,\mathbb{C}). \tag{3.65}$$

The regularities of the integral in the right hand side of (3.65) are proved in [34, Section 2.4]. As in [34, (2.81)], we have

$$d\tilde{\eta}_g(\mathcal{F},A) = \mathrm{FLI}_g(\mathcal{F}). \tag{3.66}$$

As in [34, (2.95)], the value of $\widetilde{\eta}_g(\mathcal{F},A)$ in $\Omega^*(B,\mathbb{C})/d\Omega^*(B,\mathbb{C})$ is independent of the choice of the cut-off function. Similarly, if A_P and A'_P are two smoothing operators associated with the same equivariant spectral section P, we have $\widetilde{\eta}_g(\mathcal{F},A_P) = \widetilde{\eta}_g(\mathcal{F},A'_P) \in \Omega^*(B,\mathbb{C})/d\Omega^*(B,\mathbb{C})$. In this case, we often simply denote it by $\widetilde{\eta}_g(\mathcal{F},P)$.

If the fiber Z is connected, we could calculate the equivariant eta form explicitly:

$$\tilde{\eta}_g(\mathcal{F},A) = \begin{cases} \displaystyle\int_0^\infty \frac{1}{\sqrt{\pi}}\psi_B\,\mathrm{Tr}_s^{\mathrm{even}}\left[g\,\frac{\partial\mathbb{B}'_u}{\partial u}\exp(-(\mathbb{B}'_u)^2)\right]du \in \Omega^*(B,\mathbb{C})/d\Omega^*(B,\mathbb{C}), \\ \qquad\qquad \text{if } \mathcal{F} \text{ is odd}; \\ \displaystyle\int_0^\infty \frac{1}{2\sqrt{\pi}\sqrt{-1}}\psi_B\,\mathrm{Tr}_s\left[g\,\frac{\partial\mathbb{B}'_u}{\partial u}\exp(-(\mathbb{B}'_u)^2)\right]du \in \Omega^*(B,\mathbb{C})/d\Omega^*(B,\mathbb{C}), \\ \qquad\qquad \text{if } \mathcal{F} \text{ is even and } \dim Z > 0. \\ \displaystyle\int_0^\infty \frac{\sqrt{-1}}{2\pi}\,\mathrm{Tr}_s\left[g\,\frac{\partial\nabla_u^E}{\partial u}\exp\left(-\frac{(\nabla_u^E)^2}{2\pi\sqrt{-1}}\right)\right]du \in \Omega^*(B,\mathbb{C})/d\Omega^*(B,\mathbb{C}), \\ \qquad\qquad \text{if } \dim Z = 0, \end{cases} \tag{3.67}$$

where $\nabla_u^E = \nabla^E + \sqrt{u}\chi(\sqrt{u})A$.

When $\dim Z = 0$, the equivariant geometric family degenerates to the case of Example 2.5 a). Then there exists a complex vector bundle E' such that $E_+ \oplus E' \simeq E_- \oplus E'$ as complex vector bundles. As in [42, Definition B.5.3], from (3.52) and

(3.66), the equivariant eta form in this case is just the equivariant transgression between $\mathrm{ch}_g(E_+\oplus E', \nabla^{E_+\oplus E'})$ and $\mathrm{ch}_g(E_-\oplus E', \nabla^{E_-\oplus E'})$.

Furthermore, by changing the variable (see also [34, Remark 2.4]), we could get another form of equivariant eta form:

$$\tilde{\eta}_g(\mathcal{F}, A) = -\int_0^{\infty}\left\{\psi_{\mathbb{R}\times B}\,\widetilde{\mathrm{Tr}}\left[g\exp\left(-\left(\mathbb{B}'_{u^2}+du\wedge\frac{\partial}{\partial u}\right)^2\right)\right]\right\}^{du} du. \tag{3.68}$$

Let $(Z', g^{TZ'})$ be an even dimensional Spin^c manifold and $(E', h^{E'}, \nabla^{E'})$ be a $\mathbb{Z}_2$-graded Hermitian vector bundle over Z' with a Hermitian connection $\nabla^{E'}$. Let $\mathrm{pr}_2 : B\times Z'\to Z'$ be the projection onto the second part. Then all the bundles and geometric data above could be pulled back on $B\times Z'$. Thus the fiber bundle $B\times Z'\to B$ and the structures pulled back by pr_2 form a geometric family $\mathcal{F}'$ with fibers Z'. In this case, $\mathrm{Ind}(D(\mathcal{F}'))$ is a trivial virtual complex vector bundle over B. It could also be regarded as a locally constant function on B with values in $\mathbb{Z}$. We assume that the group action on $\mathcal{F}'$ is trivial. For $\mathcal{F}\in \mathrm{F}^*_G(B)$, let A be a perturbation operator with respect to $D(\mathcal{F})$ on $\mathcal{F}$. Let τ' be the $\mathbb{Z}_2$-grading of $\mathcal{S}_{Z'}\widehat{\otimes}E'$. As in (2.7), we define

$$A\widehat{\otimes}1 := A\otimes\tau' \tag{3.69}$$

on $\mathcal{F}\times_B\mathcal{F}'$. By (2.7), we have

$$(D(\mathcal{F}\times_B\mathcal{F}')+A\widehat{\otimes}1)^2 = (D(\mathcal{F})+A)^2\widehat{\otimes}1+1\widehat{\otimes}D(\mathcal{F}')^2>0. \tag{3.70}$$

Thus $A\widehat{\otimes}1$ is a perturbation operator with respect to $D(\mathcal{F}\times_B\mathcal{F}')$.

Lemma 3.13. *For $g\in G$, we have*

$$\tilde{\eta}_g(\mathcal{F}\times_B\mathcal{F}', A\widehat{\otimes}1) = \tilde{\eta}_g(\mathcal{F}, A)\cdot\mathrm{Ind}(D(\mathcal{F}'))\in\Omega^*(B,\mathbb{C})/d\Omega^*(B,\mathbb{C}). \tag{3.71}$$

Here we consider $\mathrm{Ind}(D(\mathcal{F}'))$ *as a locally constant function on B with values in $\mathbb{Z}$.*

Proof. We denote by $\mathrm{Tr}|_{\mathcal{F}}$ the trace operator associated with $\mathcal{F}$. Then from (3.68),

$$\begin{aligned}
&\tilde{\eta}_g(\mathcal{F}\times_B\mathcal{F}', A\widehat{\otimes}1)\\
&= -\int_0^{\infty}\left\{\psi_{\mathbb{R}\times B}\,\widetilde{\mathrm{Tr}}|_{\mathcal{F}\times_B\mathcal{F}'}\left[g\exp\left(-\left(\mathbb{B}'_{u^2}\widehat{\otimes}1+1\widehat{\otimes}uD(\mathcal{F}')+du\wedge\frac{\partial}{\partial u}\right)^2\right)\right]\right\}^{du} du\\
&= \int_0^{\infty}\left\{\psi_{\mathbb{R}\times B}\,\widetilde{\mathrm{Tr}}|_{\mathcal{F}\times_B\mathcal{F}'}\left[g(1\widehat{\otimes}D(\mathcal{F}'))\exp\left(-\left(\mathbb{B}'_{u^2}\widehat{\otimes}1\right)^2-\left(1\widehat{\otimes}uD(\mathcal{F}')\right)^2\right)\right]\right\} du\\
&-\int_0^{\infty}\left\{\psi_{\mathbb{R}\times B}\,\widetilde{\mathrm{Tr}}|_{\mathcal{F}\times_B\mathcal{F}'}\left[g\exp\left(-\left(\mathbb{B}'_{u^2}\widehat{\otimes}1+du\wedge\frac{\partial}{\partial u}\right)^2-1\widehat{\otimes}u^2D(\mathcal{F}')^2\right)\right]\right\}^{du} du\\
&= \int_0^{\infty}\left\{\psi_{\mathbb{R}\times B}\,\widetilde{\mathrm{Tr}}|_{\mathcal{F}}\left[g\exp\left(-(\mathbb{B}'_{u^2})^2\right)\right]\cdot\mathrm{Tr}_s|_{\mathcal{F}'}\left[D(\mathcal{F}')\exp\left(-u^2D(\mathcal{F}')^2\right)\right]\right\} du\\
&-\int_0^{\infty}\left\{\psi_{\mathbb{R}\times B}\,\widetilde{\mathrm{Tr}}|_{\mathcal{F}}\left[g\exp\left(-\left(\mathbb{B}_{u^2}+du\wedge\frac{\partial}{\partial u}\right)^2\right)\right]\cdot\mathrm{Tr}_s|_{\mathcal{F}'}\left[\exp\left(-u^2D(\mathcal{F}')^2\right)\right]\right\}^{du} du.
\end{aligned} \tag{3.72}$$

From the definition of $\mathcal{F}'$ and the local index theorem, as functions on B, we have

$$\begin{aligned}
&\mathrm{Tr}_s|_{\mathcal{F}'}\left[D(\mathcal{F}')\exp\left(-\left(u^2D(\mathcal{F}')^2\right)\right)\right]=0,\\
&\mathrm{Tr}_s|_{\mathcal{F}'}\left[\exp\left(-u^2D(\mathcal{F}')^2\right)\right]=\mathrm{Ind}(D(\mathcal{F}')).
\end{aligned} \tag{3.73}$$

So we get $\tilde{\eta}_g(\mathcal{F}\times_B \mathcal{F}', A\widehat{\otimes}1)=\tilde{\eta}_g(\mathcal{F},A)\cdot \mathrm{Ind}(D(\mathcal{F}'))$.

The proof of Lemma 3.13 is completed. □

3.6 Anomaly formula for odd equivariant geometric families

In this subsection, we will study the anomaly formula of the equivariant eta forms for two odd equivariant geometric families $\mathcal{F}$ and $\mathcal{F}'$ with the same topological structure. In this subsection, we also assume that G acts on B trivially.

Assume that $\mathcal{F}\in \mathrm{F}^1_G(B)$. Let A be a family of bounded pseudodifferential operator on $\mathcal{F}$ such that $D(\mathcal{F})+A$ is an equivariant B-family. Let P, Q be two equivariant spectral sections with respect to $D(\mathcal{F})+A$. Let A_P, A_Q be smoothing operators associated with P, Q. Then $A+A_P$ and $A+A_Q$ are perturbation operators of $D(\mathcal{F})$. In this case, by (3.66), the difference of $\tilde{\eta}_g(\mathcal{F},A+A_P)$ and $\tilde{\eta}_g(\mathcal{F},A+A_Q)$ is closed. Furthermore, we have the following lemma.

Lemma 3.14. *(Compare with [43, Proposition 17]) For any $g\in G$, modulo exact forms on B, we have*

$$\tilde{\eta}_g(\mathcal{F},A+A_P)-\tilde{\eta}_g(\mathcal{F},A+A_Q)=\mathrm{ch}_g([P-Q])\in H^{even}(B,\mathbb{C}). \tag{3.74}$$

Proof. Note that A, A_P, A_Q preserve the $\mathbb{Z}_2$-grading of E and if we reverse the orientation of the fibers, the eta form is changed to its minus. From (3.14) and the orientation reversing trick in the proof of Proposition 3.3 (i), we only need to prove the lemma when Q majorizes P and $E_-=0$ in $\mathcal{F}$.

Let $\widetilde{\mathcal{F}}$ be the equivariant geometric family defined in (3.12) such that $\mathcal{F}_r=\mathcal{F}$ for any $r\in[0,1]$. Let $\widetilde{\mathbb{B}}_u$ be the Bismut superconnection associated with $\widetilde{\mathcal{F}}$. We choose $s>0$ large enough such that P and Q satisfy (3.1) for $f(b)\equiv s$. We choose equivariant spectral sections R' and R'' as in (3.3). Since the eta form is independent of the smoothing operators with respect to the same equivariant spectral section, we may choose the smoothing operators A_P, A_Q as in (3.3). Set $A_r:=A+rA_Q+(1-r)A_P$. Let

$$\widetilde{\mathbb{B}}'_u|_{(u,r)}:=\widetilde{\mathbb{B}}_u|_{(u,r)}+\sqrt{u}\chi(\sqrt{u})A_r \tag{3.75}$$

as in (3.62). We simply denote by

$$D_r:=D(\mathcal{F})+A_r,\quad \widetilde{\nabla}:=\nabla^{\mathcal{E}}+dr\wedge\frac{\partial}{\partial r}. \tag{3.76}$$

Then from (3.37), when $u>2$, we have

$$\begin{aligned}\left(\widetilde{\mathbb{B}}'_{u^2}\right)^2=u^2D_r^2+u[D_r,\widetilde{\nabla}]+\widetilde{\nabla}^2+\frac{1}{4}[D_r,c(T)]\\ +\frac{1}{4u}[\widetilde{\nabla},c(T)]+\frac{1}{16u^2}c(T)^2.\end{aligned} \tag{3.77}$$

For a family of bounded operators $\mathcal{A}_u$, $u\in\mathbb{R}_+$, we write $\mathcal{A}_u=O(u^{-k})$ as $u\to+\infty$ if there exists $C>0$ such that if u is large enough, the norm of $\mathcal{A}_u$ is dominated by C/u^k.

Let Π be the orthogonal projection onto $[P-Q]$, an equivariant complex vector bundle over $\widetilde{B}$. Let

$$\begin{aligned}E_u=\Pi\circ\left(\widetilde{\mathbb{B}}'_{u^2}\right)^2\circ\Pi,\quad F_u=\Pi\circ\left(\widetilde{\mathbb{B}}'_{u^2}\right)^2\circ\Pi^{\perp},\\ G_u=\Pi^{\perp}\circ\left(\widetilde{\mathbb{B}}'_{u^2}\right)^2\circ\Pi,\quad H_u=\Pi^{\perp}\circ\left(\widetilde{\mathbb{B}}'_{u^2}\right)^2\circ\Pi^{\perp}.\end{aligned} \tag{3.78}$$

From (3.3), since Π commutes with P, Q, R', R'', $\Pi Q = \Pi R' = 0$ and $\Pi R'' = \Pi P = \Pi$, we have

$$\begin{aligned}\Pi\circ D_r\circ\Pi = r\,\Pi\circ\big(R'(D(\mathcal{F})+A)R'+sQR''(I-R')+(I-R'')(D(\mathcal{F})+A)(I-R'')\\ - s(I-Q)R''(I-R')\big)\circ\Pi + (1-r)\,\Pi\circ\big(R'(D(\mathcal{F})+A)R' + sPR''(I-R')\\ + (I-R'')(D(\mathcal{F})+A)(I-R'') - s(I-P)R''(I-R')\big)\circ\Pi\\ = s(1-2r)\Pi. \quad (3.79)\end{aligned}$$

Since D_r preserves the splitting $\mathrm{Range}(\Pi)\oplus\mathrm{Range}(\Pi^{\perp})$,

$$\begin{aligned}\Pi\circ[D_r,\widetilde{\nabla}]\circ\Pi = [\Pi\circ D_r\circ\Pi,\ \Pi\circ\widetilde{\nabla}\circ\Pi]\\ = dr\wedge\frac{\partial}{\partial r}(\Pi\circ D_r\circ\Pi) = -2sdr\wedge\circ\Pi. \quad (3.80)\end{aligned}$$

Similarly, we have $\Pi\circ[D_r, c(T)]\circ\Pi = 0$ and

$$\Pi\circ\widetilde{\nabla}^2\circ\Pi = \Pi\circ(\nabla^{\mathcal{E}})^2\circ\Pi. \quad (3.81)$$

Let

$$\begin{aligned}&E' = u^2s^2(1-2r)^2\Pi - 2usdr\wedge\circ\Pi + \Pi\circ(\nabla^{\mathcal{E}})^2\circ\Pi, \quad F' = \Pi^{\perp}\circ[D_r,\widetilde{\nabla}]\circ\Pi,\\ &G' = \Pi\circ[D_r,\widetilde{\nabla}]\circ\Pi^{\perp}, \quad H' = \Pi^{\perp}\circ D_r^2\circ\Pi^{\perp}.\end{aligned} \quad (3.82)$$

From (3.77)-(3.82), when $u\to+\infty$,

$$\begin{aligned}&E_u = E' + O(u^{-1}), \quad F_u = uF' + F'' + O(1),\\ &G_u = uG' + G'' + O(1), \quad H_u = u^2H' + uH'' + H''' + O(1),\end{aligned} \quad (3.83)$$

where F'', G'', H'', H''' are first order differential operators along the fiber. Let

$$\nabla^{\Pi} := \Pi\circ\nabla^{\mathcal{E}}\circ\Pi. \quad (3.84)$$

We have

$$E' - F'H'^{-1}G' = u^2s^2(1-2r)^2\Pi - 2usdr\wedge\circ\Pi + (\nabla^{\Pi})^2. \quad (3.85)$$

Following the same way as the proof of [34, Theorem 5.13], we can obtain

$$\exp\left(-\left(\widetilde{\mathbb{B}}'_{u^2}\right)^2\right) = \Pi\circ\exp(-(E' - F'H'^{-1}G'))\circ\Pi + O(u^{-1}). \quad (3.86)$$

Thus we have

$$\left[\exp\left(-\left(\widetilde{\mathbb{B}}'_{u^2}\right)^2\right)\right]^{dr} = 2use^{-u^2s^2(1-2r)^2}\Pi\circ\exp\left(-\left(\nabla^{\Pi}\right)^2\right)\circ\Pi + O(u^{-1}). \quad (3.87)$$

Set

$$r_1(u,r) = \left\{\psi_B\,\mathrm{Tr}_s^{\mathrm{odd}}\left[g\exp\left(-\left(\widetilde{\mathbb{B}}'_{u^2}\right)^2\right)\right]\right\}^{dr}\Bigg|_{(u,r)}.$$

From [34, (2.95)], modulo exact forms on B, we have

$$\begin{aligned}\tilde{\eta}_g(\mathcal{F}, A+A_P) - \tilde{\eta}_g(\mathcal{F}, A+A_Q) = \lim_{u\to+\infty}\int_0^1 r_1(u,r)dr\\ = \frac{1}{\sqrt{\pi}}\lim_{u\to+\infty}\int_0^1 2use^{-u^2s^2(1-2r)^2}dr\cdot\psi_B\,\mathrm{Tr}[g\exp(-(\nabla^{\Pi})^2)]\\ = \frac{1}{\sqrt{\pi}}\lim_{u\to+\infty}\int_{-us}^{us}e^{-x^2}dx\cdot\mathrm{ch}_g([P-Q]) = \mathrm{ch}_g([P-Q]). \quad (3.88)\end{aligned}$$

The proof of Lemma 3.14 is completed. □

Using Lemma 3.14, we obtain the anomaly formula in odd case as follows.

Proposition 3.15. *(Compare with [20, Theorem 0.1]) Let $\mathcal{F}$, $\mathcal{F}' \in \mathrm{F}^1_G(B)$ which have the same topological structure. Let A, A' be perturbation operators with respect to $D(\mathcal{F})$, $D(\mathcal{F}')$ and P, P' be APS projections with respect to $D(\mathcal{F})+A$, $D(\mathcal{F}')+A'$ respectively. For any $g \in G$, modulo exact forms on B, we have*

$$\begin{aligned}\tilde{\eta}_g(\mathcal{F}', A') - \tilde{\eta}_g(\mathcal{F}, A) = \widetilde{\mathrm{FLI}}_g(\mathcal{F}, \mathcal{F}') \\ + \mathrm{ch}_g\left(\mathrm{sf}_G\{(D(\mathcal{F})+A, P), (D(\mathcal{F}')+A', P')\}\right).\end{aligned} \tag{3.89}$$

Proof. Let $\widetilde{\mathcal{F}}$ be the equivariant geometric family defined in (3.12). Let $D_r = D(\mathcal{F}_r) + (1-r)A + rA'$ and $\widetilde{D} = \{D_r\}_{r\in[0,1]}$ on $\widetilde{\mathcal{F}}$. Since the equivariant family index of $D(\mathcal{F})$ vanishes, so are D_r and $\widetilde{D}$. If we consider the total family $\widetilde{\mathcal{F}}$, from Proposition 3.3(i), there exists a total equivariant spectral section $\widetilde{P}$ of $\widetilde{D}$. Let P_r be the restriction of $\widetilde{P}$ over $\{r\}\times B$. Then it is an equivariant spectral section of D_r. Let A_{P_r} be an equivariant smoothing operator associated with P_r. Following the proof of [34, Theorem 2.7], we can get

$$\tilde{\eta}_g(\mathcal{F}', A' + A_{P_1}) - \tilde{\eta}_g(\mathcal{F}, A + A_{P_0}) = \widetilde{\mathrm{FLI}}_g(\mathcal{F}, \mathcal{F}'). \tag{3.90}$$

Thus Proposition 3.15 follows from Lemma 3.14, (3.15) and (3.90).

The proof of Proposition 3.15 is completed. □

3.7 Functoriality of equivariant eta forms

In this subsection, we will study the functoriality of the equivariant eta forms and use it to prove the anomaly formula of equivariant eta forms for even equivariant geometric families. In this subsection, we use the notation in Section 2.4 and assume that G acts trivially on B.

Recall that in (2.34), $TZ = T^H_{\pi_X}Z \oplus TX$. Let $\nabla^{TY,TX}$ be the connection on TZ defined by

$$\nabla^{TY,TX} = \pi_X^*\nabla^{TY} \oplus \nabla^{TX} \tag{3.91}$$

as in (3.32).

Let ∇, ∇' be Euclidean connections on (TZ, g^{TZ}) and ∇^{L_Z}, ∇'^{L_Z} be Hermitian connections on (L_Z, h^{L_Z}). Similarly as (3.50), we define

$$\mathrm{FLI}_g(\nabla, \nabla^{L_Z}) := \int_{Z^g} \mathrm{Td}_g(\nabla, \nabla^{L_Z})\, \mathrm{ch}_g(E, \nabla^E). \tag{3.92}$$

As in (3.55) and (3.56), there exists a well-defined equivariant Chern-Simons form $\widetilde{\mathrm{Td}}_g(\nabla, \nabla^{L_Z}, \nabla', \nabla'^{L_Z}) \in \Omega^*(W^g, \mathbb{C})/d\Omega^*(W^g, \mathbb{C})$ such that

$$d^{W^g}\, \widetilde{\mathrm{Td}}_g(\nabla, \nabla^{L_Z}, \nabla', \nabla'^{L_Z}) = \mathrm{Td}_g(\nabla', \nabla'^{L_Z}) - \mathrm{Td}_g(\nabla, \nabla^{L_Z}). \tag{3.93}$$

Set

$$\widetilde{\mathrm{FLI}}_g(\nabla, \nabla^{L_Z}, \nabla', \nabla'^{L_Z}) := \int_{Z^g} \widetilde{\mathrm{Td}}_g(\nabla, \nabla^{L_Z}, \nabla', \nabla'^{L_Z})\, \mathrm{ch}_g(E, \nabla^E). \tag{3.94}$$

From (3.58) and (3.93), we have

$$d^B\, \widetilde{\mathrm{FLI}}_g(\nabla, \nabla^{L_Z}, \nabla', \nabla'^{L_Z}) = \mathrm{FLI}_g(\nabla', \nabla'^{L_Z}) - \mathrm{FLI}_g(\nabla, \nabla^{L_Z}). \tag{3.95}$$

From the proof of Lemma 3.6, we obtain that if A_X is a perturbation operator of $D(\mathcal{F}_X)$, there exists $T' > 0$ such that when $T \geqslant T'$, $1\widehat{\otimes}TA_X$ is a perturbation operator of $D(\mathcal{F}_{Z,T})$.

Note that when A_X is a family of smoothing operators along the fibers X, $1\widehat{\otimes}A_X$ is only bounded, not a family of smoothing operators along the fibers Z. This is the reason for us to define the eta form for the bounded perturbation operator instead of the smoothing operator in [14, 16, 20, 43].

The following technical lemma is a modification of the main result in [34]. The proof of it will be left to the next subsection.

Lemma 3.16. *Modulo exact forms on B, for $T \geqslant T'$, we have*

$$\widetilde{\eta}_g(\mathcal{F}_{Z,T}, 1\widehat{\otimes}TA_X) = \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y})\,\widetilde{\eta}_g(\mathcal{F}_X, A_X) + \widetilde{\mathrm{FLI}}_g\left(\nabla^{TY,TX}, \nabla^{L_Z}, \nabla^{TZ}_T, \nabla^{L_Z}\right). \quad (3.96)$$

Here ∇^{TZ}_T is the connection associated with $(T^H_{\pi_Z}W, g^{TZ}_T)$ as in (2.14).

Using Lemma 3.16, we could extend the anomaly formula Proposition 3.15 to the general case.

Theorem 3.17. *Let $\mathcal{F}$, $\mathcal{F}' \in \mathrm{F}^*_G(B)$ which have the same topological structure. Let A, A' be perturbation operators with respect to $D(\mathcal{F})$, $D(\mathcal{F}')$ and P, P' be the APS projections with respect to $D(\mathcal{F})+A$, $D(\mathcal{F}')+A'$ respectively. For any $g \in G$, modulo exact forms on B, we have*

$$\tilde{\eta}_g(\mathcal{F}', A') - \tilde{\eta}_g(\mathcal{F}, A) = \widetilde{\mathrm{FLI}}_g(\mathcal{F}, \mathcal{F}') + \mathrm{ch}_g\left(\mathrm{sf}_G\{(D(\mathcal{F})+A, P), (D(\mathcal{F}')+A', P')\}\right). \quad (3.97)$$

Proof. We only need to prove the even case.

Let $L \to S^1 \times S^1$ be the Hermitian line bundle in Example 2.5 c) with ∇^L constructed there. We use the notation in Example 2.5 c). Let $p : B \times S^1 \times S^1 \to S^1 \times S^1$ be the natural projection. Then all bundles and geometric data in $\mathcal{F}^L$ could be pulled back on $B \times S^1 \times S^1$. Thus the fiber bundle $B \times S^1 \times S^1 \to B$ and the structures pulled back by p form an even geometric family $\mathcal{F}_0$ over B. In this case, $\mathrm{Ind}(D(\mathcal{F}_0)) = 1$. Here we consider $\mathrm{Ind}(D(\mathcal{F}_0))$ as a locally constant function on B as in Lemma 3.13. The key observation is

$$p_1!(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L) = \mathcal{F} \times_B \mathcal{F}_0. \quad (3.98)$$

Recall that $A\widehat{\otimes}1_{\mathcal{F}_0}$ is defined in (3.69). Since $A\widehat{\otimes}1_{\mathcal{F}_0}$ is a perturbation operator of $D(\mathcal{F}\times_B \mathcal{F}_0)$, we could choose $T' = 1$ in Lemma 3.16. By Lemmas 3.13 and 3.16, we have

$$\begin{aligned}\tilde{\eta}_g(\mathcal{F}, A) &= \tilde{\eta}_g(\mathcal{F}\times_B \mathcal{F}_0, A\widehat{\otimes}1_{\mathcal{F}_0}) \\ &= \int_{S^1} \tilde{\eta}_g(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L, A\widehat{\otimes}1_{\mathcal{F}^L}) \\ &\quad - \widetilde{\mathrm{FLI}}_g\left(\nabla^{T(Z\times S^1)}, \nabla^{L_Z}, \nabla^{TZ,TS^1}, \nabla^{L_Z}\right). \end{aligned} \quad (3.99)$$

As in (3.16), $D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L) = D(\mathcal{F}) \otimes 1 + \tau \otimes D^L$. By Proposition 3.15, the construction of the equivariant higher spectral flow for even case and (3.43), we have

$$\begin{aligned}&\tilde{\eta}_g(\mathcal{F}', A') - \tilde{\eta}_g(\mathcal{F}, A) \\ &= \int_{S^1} \left\{\tilde{\eta}_g(p_1^*\mathcal{F}' \times_{B\times S^1} p_2^*\mathcal{F}^L, A'\widehat{\otimes}1_{\mathcal{F}^L}) - \tilde{\eta}_g(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L, A\widehat{\otimes}1_{\mathcal{F}^L})\right\} \\ &\quad + \int_{S^1}\int_{Z^g} \left\{\widetilde{\mathrm{Td}}_g\left(\nabla^{T(S^1\times Z')}, \nabla^{L_{Z'}}, \nabla^{TS^1,TZ'}, \nabla^{L_{Z'}}\right) \mathrm{ch}_g(E', \nabla^{E'})\right.\end{aligned}$$

$$
\begin{aligned}
& - \widetilde{\mathrm{Td}}_g \left(\nabla^{T(S^1\times Z)}, \nabla^{L_Z}, \nabla^{TS^1,TZ}, \nabla^{L_Z}\right) \mathrm{ch}_g(E,\nabla^E)\Big\} \\
&= \widetilde{\mathrm{FLI}}_g(\mathcal{F},\mathcal{F}') + \int_{S^1} \mathrm{ch}_g \left(\mathrm{sf}_G\{(D(p_1^*\mathcal{F}\times_{B\times S^1} p_2^*\mathcal{F}^L) + A\widehat{\otimes}1, P_0),\right. \\
&\qquad \left.(D(p_1^*\mathcal{F}'\times_{B\times S^1} p_2^*\mathcal{F}^L) + A'\widehat{\otimes}1, P_0')\}\right) \\
&= \widetilde{\mathrm{FLI}}_g(\mathcal{F},\mathcal{F}') + \mathrm{ch}_g\left(\mathrm{sf}_G\{(D(\mathcal{F})+A,P),(D(\mathcal{F}')+A',P')\}\right), \quad (3.100)
\end{aligned}
$$

where P_0, P_0' are the associated APS projections respectively. Note that in order to adapt the sign convention (1.7), the sign in the beginning of the fifth line of (3.100) is alternated.

The proof of Theorem 3.17 is completed. □

Using Theorem 3.17, we could write Lemma 3.16 as a more elegant form.

Theorem 3.18. *Let A_Z and A_X be perturbation operators with respect to $D(\mathcal{F}_Z)$ and $D(\mathcal{F}_X)$. Then modulo exact forms on B, for $T\geqslant 1$ large enough, we have*

$$
\begin{aligned}
\widetilde{\eta}_g(\mathcal{F}_Z,A_Z) &= \int_{Y^g} \mathrm{Td}_g(\nabla^{TY},\nabla^{L_Y})\,\widetilde{\eta}_g(\mathcal{F}_X,A_X) + \widetilde{\mathrm{FLI}}_g\left(\nabla^{TY,TX},\nabla^{L_Z},\nabla^{TZ},\nabla^{L_Z}\right) \\
&\quad + \mathrm{ch}_g(\mathrm{sf}_G\{(D(\mathcal{F}_{Z,T})+1\widehat{\otimes}TA_X,P),(D(\mathcal{F}_Z)+A_Z,P')\}), \quad (3.101)
\end{aligned}
$$

where P and P' are the associated APS projections respectively.

From Theorems 3.17 and 3.18, we could extend Lemma 3.13 to the general case.

Theorem 3.19. *(Compare with [16, (24)]) Let $\mathcal{F},\mathcal{F}'\in \mathrm{F}_G^*(B)$. Let A and A' be the perturbation operators with respect to $D(\mathcal{F})$ and $D(\mathcal{F}\times_B\mathcal{F}')$. Then there exists $x\in K_G^*(B)$, such that*

$$
\widetilde{\eta}_g(\mathcal{F}\times_B\mathcal{F}',A') = \widetilde{\eta}_g(\mathcal{F},A)\,\mathrm{FLI}_g(\mathcal{F}') + \mathrm{ch}_g(x). \quad (3.102)
$$

Proof. Here we use a trick in [16] similarly as (3.98). Let $\pi' : W'\to B$ be the submersion in $\mathcal{F}'$. We could obtain a pullback family $\pi'^*\mathcal{F}$ by choosing a horizontal subbundle $T^H(\pi'^*W)$ such that $d\pi'(T^H(\pi'^*W))\subset T^HW$. Let $\pi'^*\mathcal{F}\otimes E'$ be the equivariant geometric family which is obtained from $\pi'^*\mathcal{F}$ by twisting with $P_{W'}^*(\mathcal{S}_{Z'}\otimes E')$, where $P_{W'}: W\times_B W'\to W'$. Then we have

$$
\mathcal{F}\times_B\mathcal{F}' \simeq \pi'!(\pi'^*\mathcal{F}\otimes E'). \quad (3.103)
$$

Since the fibers of $\pi'^*W\to B$ is $Z'\times Z$, the fiberwise connection $\nabla^{T(Z'\times Z)} = \nabla^{TZ',TZ}$. So Theorem 3.19 follows from Theorem 3.18.

The proof of Theorem 3.19 is completed. □

Remark 3.20. When the parameter space B is a point and $\dim Z$ is odd, letting $A=P_{\ker D}$ be the orthogonal projection onto the kernel of $D(\mathcal{F})$, which we simply denote by D, we have

$$
\begin{aligned}
\tilde{\eta}_g(\mathcal{F},A) &= \frac{1}{\sqrt{\pi}}\int_0^{+\infty} \mathrm{Tr}\left[g(D+(u\chi(u))'P_{\ker D})\exp(-(uD+u\chi(u)P_{\ker D})^2)\right]du \\
&= \frac{1}{\sqrt{\pi}}\int_0^{+\infty} \mathrm{Tr}\left[g(D+(u\chi(u))'P_{\ker D})\exp(-u^2D-u^2\chi(u)^2P_{\ker D})\right]du \\
&= \frac{1}{\sqrt{\pi}}\int_0^{+\infty} \mathrm{Tr}\left[gD\exp(-u^2D^2)\right]du \\
&\quad + \frac{1}{\sqrt{\pi}}\int_0^{+\infty} \mathrm{Tr}\left[g(u\chi(u))'P_{\ker D}\exp(-u^2\chi(u)^2P_{\ker D})\right]du
\end{aligned}
$$

$$= \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} u^{-1/2}\,\mathrm{Tr}[gD\exp(-uD^2)]du + \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \exp(-u^2)du \cdot \mathrm{Tr}[gP_{\ker D}]$$

$$= \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} u^{-1/2}\,\mathrm{Tr}[gD\exp(-uD^2)]du + \frac{1}{2}\,\mathrm{Tr}[gP_{\ker D}], \quad (3.104)$$

which is just the usual **equivariant reduced eta invariant** in [22]. So Theorem 3.18 naturally degenerates to the case of equivariant reduced eta invariants and the equivariant higher spectral flow degenerates to the canonical equivariant spectral flow [24].

3.8 Proof of Lemma 3.16

The proof of Lemma 3.16 is almost the same as the proof of [34, Theorem 3.4]. Observe that Assumptions 3.1 and 3.3 in [34] naturally hold in our case.

Let $T' \geqslant 1$ be the constant taking in the proof of Lemma 3.6. For $T \geqslant T'$, let $\mathbb{B}_{u,T}$ be the Bismut superconnection associated with the equivariant geometric family $\mathcal{F}_{Z,T}$. Let

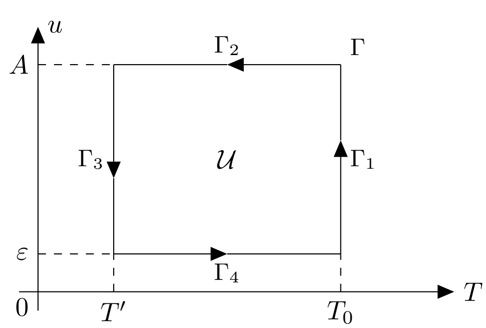
$$\widehat{\mathbb{B}}|_{(T,u)} = \mathbb{B}_{u^2,T} + uT\chi(uT)(1\widehat{\otimes}A_X) + dT \wedge \frac{\partial}{\partial T} + du \wedge \frac{\partial}{\partial u}. \quad (3.105)$$

We define $\beta_g = du \wedge \beta_g^u + dT \wedge \beta_g^T$ to be the part of $\psi_B\widetilde{\mathrm{Tr}}[g\exp(-\widehat{\mathbb{B}}^2)]$ of degree one with respect to the coordinates (T,u), with functions β_g^u, $\beta_g^T : \mathbb{R}_{+,T} \times \mathbb{R}_{+,u} \to \Omega^*(B,\mathbb{C})$.

Comparing with [34, Proposition 4.2], there exists a smooth family $\alpha_g : \mathbb{R}_{+,T} \times \mathbb{R}_{+,u} \to \Omega^*(B,\mathbb{C})$ such that

$$\left(du \wedge \frac{\partial}{\partial u} + dT \wedge \frac{\partial}{\partial T}\right)\beta_g = dT \wedge du \wedge d^B\alpha_g. \quad (3.106)$$

Take ε, A, T_0, $0 < \varepsilon \leqslant 1 \leqslant A < \infty$, $T' \leqslant T_0 < \infty$. Let $\Gamma = \Gamma_{\varepsilon,A,T_0}$ be the oriented contour in $\mathbb{R}_{+,T} \times \mathbb{R}_{+,u}$.



The contour Γ is made of four oriented pieces $\Gamma_1, \cdots, \Gamma_4$ indicated in the above picture. For $1 \leqslant k \leqslant 4$, set $I_k^0 = \int_{\Gamma_k} \beta_g$. Then by Stocks' formula and (3.106),

$$\sum_{k=1}^{4} I_k^0 = \int_{\partial\mathcal{U}} \beta_g = \int_{\mathcal{U}} \left(du \wedge \frac{\partial}{\partial u} + dT \wedge \frac{\partial}{\partial T}\right)\beta_g = d^B\left(\int_{\mathcal{U}} \alpha_g dT \wedge du\right). \quad (3.107)$$

The following theorems are the analogues of [34, Theorems 4.3-4.6]. Note that Theorem 3.22 is the analogue of [34, (6.8)]. We will sketch the proofs in the next subsection.

Theorem 3.21. *i) For any $u > 0$, we have*

$$\lim_{T\to\infty} \beta_g^u(T,u) = 0. \tag{3.108}$$

ii) For $0 < u_1 < u_2$ fixed, there exists $C > 0$ such that, for $u \in [u_1, u_2]$, $T \geqslant 1$, we have

$$|\beta_g^u(T,u)| \leqslant C. \tag{3.109}$$

iii) We have the following identity:

$$\lim_{T\to+\infty} \int_1^{\infty} \beta_g^u(T,u)du = 0. \tag{3.110}$$

Theorem 3.22. *For $u_0 > 0$ fixed, there exist $C, C' > 0$, $T_0 \geqslant 1$, such that for $u \geqslant u_0$, $T \geqslant T_0$,*

$$\left|\beta_g^T(T,u)\right| \leqslant C\exp(-C'u^2). \tag{3.111}$$

We know that $\widehat{\mathrm{A}}_g(TZ,\nabla)$ only depends on $g \in G$ and $R := \nabla^2$. So we also denote it by $\widehat{\mathrm{A}}_g(R)$. Let $R_T^{TZ} := (\nabla_T^{TZ})^2$. Set

$$\gamma_\Omega(T) = -\left.\frac{\partial}{\partial b}\right|_{b=0} \widehat{\mathrm{A}}_g\left(R_T^{TZ} + b\frac{\partial\nabla_T^{TZ}}{\partial T}\right). \tag{3.112}$$

By a standard argument in Chern-Weil theory, we know that

$$\frac{\partial}{\partial T}\widetilde{\widehat{\mathrm{A}}}_g(TZ, \nabla_{T'}^{TZ}, \nabla_T^{TZ}) = -\gamma_\Omega(T). \tag{3.113}$$

Theorem 3.23. *When $T \to +\infty$, we have $\gamma_\Omega(T) = O(T^{-2})$. Moreover, modulo exact forms on W^g, we have*

$$\widetilde{\widehat{\mathrm{A}}}_g(TZ, \nabla_{T'}^{TZ}, \nabla^{TY,TX}) = -\int_{T'}^{+\infty} \gamma_\Omega(T)dT. \tag{3.114}$$

Let $\mathbb{B}_{X,T}$ be the Bismut superconnection associated with the equivariant geometric family $\mathcal{F}_{X,T}$, which is the same as $\mathcal{F}_X$ except for replacing g^{TX} to $T^{-2}g^{TX}$. Set

$$\begin{aligned}\gamma_1(T) = \Big\{\psi_{V^g}\widetilde{\mathrm{Tr}}|_{V^g}\big[g\exp\big(-\big(\mathbb{B}_{X,T^2}|_{V^g} \\ +T\,\chi(T)A_X|_{V^g} + dT\wedge\frac{\partial}{\partial T}\Big)^2\Big)\Big]\Big\}^{dT}.\end{aligned} \tag{3.115}$$

Then from (3.68),

$$\widetilde{\eta}_g(\mathcal{F}_X, A_X) = -\int_0^{\infty} \gamma_1(T)dT. \tag{3.116}$$

Theorem 3.24. *i) For any $u > 0$, there exist $C > 0$ and $\delta > 0$ such that, for $T \geqslant T'$, we have*

$$|\beta_g^T(T,u)| \leqslant \frac{C}{T^{1+\delta}}. \tag{3.117}$$

ii) For any $T > 0$, we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\beta_g^T(T\varepsilon^{-1},\varepsilon) = \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y})\,\gamma_1(T). \tag{3.118}$$

iii) There exists $C>0$ such that for $\varepsilon\in(0,1/T']$, $\varepsilon T'\leqslant T\leqslant 1$,

$$\varepsilon^{-1}\left|\beta_g^T(T\varepsilon^{-1},\varepsilon)+\int_{Z^g}\gamma_\Omega(T\varepsilon^{-1})\ \mathrm{ch}_g(L_Y^{1/2},\nabla^{L_Y^{1/2}})\ \mathrm{ch}_g(E,\nabla^E)\right|\leqslant C. \tag{3.119}$$

Note that as in (3.46), $\mathrm{ch}_g(L_Y^{1/2},\nabla^{L_Y^{1/2}})$ is well-defined even if $L_Y^{1/2}$ does not exist.

iv) There exist $\delta\in(0,1]$, $C>0$ such that, for $\varepsilon\in(0,1]$, $T\geqslant 1$,

$$\varepsilon^{-1}|\beta_g^T(T\varepsilon^{-1},\varepsilon)|\leqslant\frac{C}{T^{1+\delta}}. \tag{3.120}$$

Now we prove Lemma 3.16 using the theorems above.

By (3.107), we know that

$$\int_\varepsilon^A\beta_g^u(T_0,u)du-\int_{T'}^{T_0}\beta_g^T(T,A)dT-\int_\varepsilon^A\beta_g^u(T',u)du +\int_{T'}^{T_0}\beta_g^T(T,\varepsilon)dT=\sum_{k=0}^4 I_k^0 \tag{3.121}$$

is an exact form. We take the limits $A\to+\infty$, $T_0\to+\infty$ and then $\varepsilon\to 0$ in the indicated order. Let I_j^k, $j=1,2,3,4$, $k=1,2,3$, denote the value of the part I_j after the kth limit.

Since the definition of the equivariant eta form does not depend on the cut-off function, from (3.65), we obtain that modulo exact forms on B,

$$I_3^3=\tilde{\eta}_g(\mathcal{F}_{Z,T'},1\widehat{\otimes}T'A_X). \tag{3.122}$$

Furthermore, by Theorem 3.22, we get

$$I_2^3=I_2^2=0. \tag{3.123}$$

From Theorem 3.21, we have

$$I_1^3=0. \tag{3.124}$$

Finally, using Theorem 3.24, we get

$$I_4^3=-\int_{Y^g}\mathrm{Td}_g(\nabla^{TY},\nabla^{L_Y})\,\widetilde{\eta}_g(\mathcal{F}_X,A_X) +\widetilde{\mathrm{FLI}}_g\left(\nabla_{T'}^{TZ},\nabla^{L_Z},\nabla^{TY,TX},\nabla^{L_Z}\right) \tag{3.125}$$

as follows: We write

$$\int_{T'}^{+\infty}\beta_g^T(T,\varepsilon)dT=\int_{\varepsilon T'}^{+\infty}\varepsilon^{-1}\beta_g^T(T\varepsilon^{-1},\varepsilon)dT. \tag{3.126}$$

Convergence of the integrals above is guaranteed by (3.117). Using Theorem 3.23 and (3.118)-(3.120), we get

$$\lim_{\varepsilon\to0}\int_1^{+\infty}\varepsilon^{-1}\beta_g^T(T\varepsilon^{-1},\varepsilon)dT=\int_{Y^g}\mathrm{Td}_g(\nabla^{TY},\nabla^{L_Y})\int_1^{+\infty}\gamma_1(T)dT \tag{3.127}$$

and

$$\lim_{\varepsilon\to0}\int_{\varepsilon T'}^1\varepsilon^{-1}\left[\beta_g^T(T\varepsilon^{-1},\varepsilon)dT+\int_{Z^g}\gamma_\Omega(T\varepsilon^{-1})\ \mathrm{ch}_g(L_Y^{1/2},\nabla^{L_Y^{1/2}})\ \mathrm{ch}_g(E,\nabla^E)\right]dT$$

$$= \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y}) \int_0^1 \gamma_1(T)dT. \quad (3.128)$$

The remaining part of the integral yields by (3.119)

$$\lim_{\varepsilon\to 0} \int_{\varepsilon T'}^1 \varepsilon^{-1} \int_{Z^g} \gamma_\Omega(T\varepsilon^{-1}) \operatorname{ch}_g(L_Y^{1/2}, \nabla^{L_Y^{1/2}}) \operatorname{ch}_g(E, \nabla^E)dT$$
$$= \int_{Z^g} \int_{T'}^{+\infty} \gamma_\Omega(T) \operatorname{ch}_g(L_Y^{1/2}, \nabla^{L_Y^{1/2}}) \operatorname{ch}_g(E, \nabla^E)dT$$
$$= \widetilde{\mathrm{FLI}}_g\left(\nabla^{TY,TX}, \nabla^{L_Z}, \nabla^{TZ}_{T'}, \nabla^{L_Z}\right). \quad (3.129)$$

These four equations for I_k^3, $k = 1, 2, 3, 4$, (3.107) and (3.121) imply that

$$\lim_{\varepsilon\to 0} \lim_{T_0\to+\infty} \lim_{A\to+\infty} d^B \left(\int_{T'}^{T_0} \int_{\varepsilon}^{A} \alpha_g dT \wedge du \right) \quad (3.130)$$

exists and equal to

$$\Theta := \widetilde{\eta}_g(\mathcal{F}_{Z,T'}, 1\widehat{\otimes} TA_X) - \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y})\, \widetilde{\eta}_g(\mathcal{F}_X, A_X)$$
$$- \widetilde{\mathrm{FLI}}_g\left(\nabla^{TY,TX}, \nabla^{L_Z}, \nabla^{TZ}_{T'}, \nabla^{L_Z}\right) \in \Omega^*(B). \quad (3.131)$$

Since the convergences for $A \to +\infty$, $T_0 \to +\infty$ and $\varepsilon \to 0$ are uniform on compact manifold B, they commute with the integration on B. So for any closed form $\theta \in \Omega^*(B)$, $\int_B \Theta \wedge \theta = 0$. By [23, §22, Theorem 17'], there exists a current $\mathcal{T}$ such that $\Theta = d\mathcal{T}$. Since $\Theta \in \Omega^*(B)$ is smooth, we have $\Theta \in d\Omega^*(B)$. So the right hand side of (3.131) is an exact form on B. Therefore we obtain Lemma 3.16.

3.9 Proofs of Theorems 3.21-3.24

Since $\ker(D(\mathcal{F}_X)+A_X) = 0$, the proofs of Theorems 3.21-3.24 in our case are much easier than those in [34]. We only need to replace D^X and D^Z_T somewhere in [34] by $D(\mathcal{F}_X)+A_X$ and $D(\mathcal{F}_{Z,T})+1\widehat{\otimes} TA_X$ and take care with the local index computation in the proof of Theorem 3.24 ii). In this subsection, we only sketch the local index part here.

Set (cf. [34, (7.1)])

$$\mathcal{B}'_{\varepsilon,T/\varepsilon} = (\mathbb{B}_{\varepsilon^2,T/\varepsilon} + T\chi(T)A_X)^2$$
$$+ \varepsilon^{-1}dT \wedge \left.\frac{\partial(\mathbb{B}_{\varepsilon^2,T'} + \varepsilon T'\chi(\varepsilon T')A_X)}{\partial T'}\right|_{T'=T\varepsilon^{-1}}. \quad (3.132)$$

By the definition of $\beta_g^T(T,\varepsilon)$, we have

$$\varepsilon^{-1}\beta_g^T(T/\varepsilon, \varepsilon) = \left\{ \psi_B \widetilde{\mathrm{Tr}}[g \exp(-\mathcal{B}'_{\varepsilon,T/\varepsilon})] \right\}^{dT}. \quad (3.133)$$

Let S_X be the tensor in (3.33) with respect to π_X. Let $\{e_i\}$, $\{f_p\}$ and $\{g_\alpha\}$ be the local orthonormal frames of TX, TY and TB and $\{f^H_{p,1}\}$ and $\{g^H_{\alpha,3}\}$ be the corresponding horizontal lifts. Precisely, by (3.37), we have

$$\varepsilon^{-1} \left.\frac{\partial(\mathbb{B}_{\varepsilon^2,T'} + \varepsilon T'\chi(\varepsilon T')A_X)}{\partial T'}\right|_{T'=T\varepsilon^{-1}} = D^X + \chi(T)A_X + T\chi'(T)A_X$$
$$- \frac{1}{8T^2}\big(\langle \varepsilon^2 [f^H_{p,1}, f^H_{q,1}], e_i\rangle c(e_i)c(f^H_{p,1})c(f^H_{q,1})$$

$$+ 4\varepsilon\langle S_X(g^H_{\alpha,3})e_i, f^H_{p,1}\rangle c(e_i)c(f^H_{p,1})g^\alpha_3 \wedge + \langle [g^H_{\alpha,3}, g^H_{\beta,3}], e_i\rangle c(e_i) g^\alpha \wedge g^\beta \wedge \Big). \quad (3.134)$$

As in (3.132), we set

$$\mathcal{B}''_{T^2}|_{V^g} = (\mathbb{B}_{X,T^2}|_{V^g} + T\chi(T)A_X)^2 + dT \wedge \left.\frac{\partial(\mathbb{B}_{X,T^2} + T\chi(T)A_X)}{\partial T}\right|_{V^g}. \quad (3.135)$$

Then by (3.115), we have

$$\gamma_1(T) = \left\{\psi_{V^g}\widetilde{\mathrm{Tr}}[g\exp(-\mathcal{B}''_{T^2}|_{V^g})]\right\}^{dT}. \quad (3.136)$$

As the same process in [34, Section 7], we could localize the problem near $\pi_X^{-1}(V^g)$ and define the operator $\mathcal{B}'_{\varepsilon,T/\varepsilon}$ on a neighborhood of $\{0\} \times X_{y_0}$ in $T_{y_0}Y \times X_{y_0}$.

Let dist^V, dist^W be the distance functions on V, W associated with g^{TV}, g^{TW}. Let Inj^V, Inj^W be the injective radius of V, W. In the sequel, we assume that given $0 < \alpha < \alpha_0 < \inf\{\mathrm{Inj}^V, \mathrm{Inj}^W\}$ are chosen small enough so that if $y \in V$, $\mathrm{dist}^V(g^{-1}y, y) \leqslant \alpha$, then $\mathrm{dist}^V(y, V^g) \leqslant \frac{1}{4}\alpha_0$, and if $z \in W$, $\mathrm{dist}^W(g^{-1}z, z) \leqslant \alpha$, then $\mathrm{dist}^W(z, W^g) \leqslant \frac{1}{4}\alpha_0$. Let $\rho : T_{y_0}Y \to [0,1]$ be a smooth function such that

$$\rho(U) = \begin{cases} 1, \ |U| \leqslant \alpha_0/4; \\ 0, \ |U| \geqslant \alpha_0/2. \end{cases} \quad (3.137)$$

Let Δ^{TY} be the ordinary Laplacian operator on $T_{y_0}Y$. Let $\mathcal{E}_{Z,y_0} := \mathcal{C}^\infty(X_{y_0}, \mathcal{S}_Z\widehat{\otimes}E|_{X_{y_0}})$.

Set

$$L^1_{\varepsilon,T} = (1-\rho^2(U))(-\varepsilon^2\Delta^{TY} + T^2(D^X + A_X)^2_{y_0}) + \rho^2(U)\mathcal{B}'_{\varepsilon,T/\varepsilon} \quad (3.138)$$

on $\pi_2^*\Lambda(T^*S)\widehat{\otimes}\mathcal{C}^\infty(T_{y_0}Y, \mathcal{E}_{Z,y_0})$. For $(U,x) \in N_{Y^g/Y,y_0} \times X_{y_0}$, $|U| < \alpha_0/4$, $\varepsilon > 0$, set

$$(S_\varepsilon s)(U,x) = s\left(U/\varepsilon, x\right). \quad (3.139)$$

Put

$$L^2_{\varepsilon,T} := S_\varepsilon^{-1} L^1_{\varepsilon,T} S_\varepsilon. \quad (3.140)$$

Set $\dim T_{y_0}Y^g = l'$ and $\dim N_{Y^g/Y,y_0} = 2l''$. Let $\{f_1, \cdots, f_{l'}\}$ be an orthonormal basis of $T_{y_0}Y^g$ and let $\{f_{l'+1}, \cdots, f_{l'+2l''}\}$ be an orthonormal basis of $N_{Y^g/Y,y_0}$. Let R_ε be a rescaling operator such that

$$\begin{aligned} &R_\varepsilon(c(e_i)) = c(e_i), \\ &R_\varepsilon(c(f^H_{p,1})) = \frac{f^p\wedge}{\varepsilon} - \varepsilon\, i_{f_p}, \quad \text{for } 1 \leqslant p \leqslant l', \\ &R_\varepsilon(c(f^H_{p,1})) = c(f^H_{p,1}), \quad \text{for } l'+1 \leqslant p \leqslant l'+2l''. \end{aligned} \quad (3.141)$$

Then R_ε is a Clifford algebra homomorphism. Set

$$L^3_{\varepsilon,T} = R_\varepsilon(L^2_{\varepsilon,T}) \quad (3.142)$$

on $\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*_{y_0}Y^g)\widehat{\otimes}\mathcal{C}^\infty(T_{y_0}Y, \mathcal{E}_{X,N,y_0})$, where $\mathcal{E}_{X,N,y_0} := \mathcal{C}^\infty(X_{y_0}, \pi_2^*\mathcal{S}_N\widehat{\otimes}\mathcal{S}_X\widehat{\otimes}E|_{X_{y_0}})$ and $\mathcal{S}_N$ is the spinor for $N_{Y^g/Y,y_0}$.

Corresponding to [34, Lemma 4.4], from (3.133)-(3.135), we have

Lemma 3.25. *When $\varepsilon \to 0$, the limit $L^3_{0,T} = \lim_{\varepsilon\to 0} L^3_{\varepsilon,T}$ exists in the sense of [34, (7.108)] and*

$$L^3_{0,T}|_{V^g} = -\left(\partial_p + \frac{1}{4}\langle R^{TY}|_{V^g}U, f^H_{p,1}\rangle\right)^2 + \frac{1}{2}R^{L_Y}|_{V^g} + \mathcal{B}''_{T^2}|_{V^g}. \quad (3.143)$$

So all computations in our case are the same as [34, Section 7].

4 Equivariant differential K-theory

In this section, we assume that the G-action on B has finite stabilizers only, i.e., for any $b \in B$, $G_b := \{g \in G : gb = b\}$ is finite. With this action, we construct an analytic model of equivariant differential K-theory and prove some properties using the results in Section 3.

4.1 Definition of equivariant differential K-theory

In this subsection, we construct an analytic model of equivariant differential K-theory. When $G = \{e\}$, this construction is similar as that in [16] except replacing the taming and KK-theory to the spectral section and higher spectral flow.

Let E be a G-equivariant complex vector bundle over B. Then its restriction to B^g is acted on fibrewise by g for $g \in G$. So it decomposes as a direct sum of subbundles E_v for each eigenvalue v of g. Set $\phi_g(E) := \sum vE_v$. Then it induces a homomorphism (for K_G^1, replacing B by $B \times S^1$ and use (2.19))

$$\phi_g : K_G^*(B) \otimes \mathbb{C} \longrightarrow [K^*(B^g) \otimes \mathbb{C}]^{\mathrm{C}_G(g)}, \tag{4.1}$$

where $\mathrm{C}_G(g)$ is the centralizer of g in G. Let (g) be the conjugacy class of $g \in G$. For $g, g' \in (g)$, there exists $h \in G$, such that $g' = h^{-1}gh$. Furthermore, the map

$$h : B^{g'}/\mathrm{C}_G(g') \to B^g/\mathrm{C}_G(g) \tag{4.2}$$

is a homeomorphism . So

$$[K^*(B^g) \otimes \mathbb{C}]^{\mathrm{C}_G(g)} \simeq [K^*(B^{g'}) \otimes \mathbb{C}]^{\mathrm{C}_G(g')}. \tag{4.3}$$

By [1, Corollary 3.13], we know that the additive decomposition

$$\phi = \oplus_{(g),g\in G}\phi_g : K_G^*(B) \otimes \mathbb{C} \to \bigoplus_{(g),g\in G} [K^*(B^g) \otimes \mathbb{C}]^{\mathrm{C}_G(g)} \tag{4.4}$$

is an isomorphism, where (g) ranges over the conjugacy classes of G.

If $B^g \neq \emptyset$, then there exists $b \in B^g$ such that $g \in G_b$. The conjugacy class of G_b is the type of the orbit $G \cdot b$. Since B is compact, there are only finitely many orbit types. Since all stablizers are finite groups, we see that the direct sum in (4.4) only has finite terms. From the isomorphism (4.3), the direct sum in (4.4) does not depend on the choice of the element in (g) in the sense of (4.3).

From (4.2), we also know that the map h^* induces an isomorphism

$$h^* : [\Omega^*(B^g, \mathbb{C})]^{\mathrm{C}_G(g)} \to [\Omega^*(B^{g'}, \mathbb{C})]^{\mathrm{C}_G(g')}. \tag{4.5}$$

We denote by

$$\Omega^*_{deloc,G}(B, \mathbb{C}) := \bigoplus_{(g),g\in G} \left\{[\Omega^*(B^g, \mathbb{C})]^{\mathrm{C}_G(g)}\right\}, \tag{4.6}$$

the set of delocalized differential forms, where $\{\cdot\}$ denotes the isomorphic class in sense of (4.5). The definition above does not depend on the choice of $g \in (g)$. It is easy to see that the exterior differential operator d preserves $\Omega^*_{deloc,G}(B, \mathbb{C})$. We denote by the delocalized de Rham cohomology $H^*_{deloc,G}(B, \mathbb{C})$ the cohomology of the differential complex $(\Omega^*_{deloc,G}(B, \mathbb{C}), d)$. Then from (4.1) and (4.4), the equivariant Chern character isomorphism can be naturally defined by

$$\begin{aligned} \mathrm{ch}_G : K_G^*(B) \otimes \mathbb{C} &\xrightarrow{\simeq} H^*_{deloc,G}(B, \mathbb{C}), \\ \mathcal{K} \quad &\mapsto \bigoplus_{(g),g\in G} \{\mathrm{ch}(\phi_g(\mathcal{K}))\}. \end{aligned} \tag{4.7}$$

We note that $\mathrm{ch}(\phi_g(\mathcal{K})) = \mathrm{ch}_g(\mathcal{K})$ is $\mathrm{C}_G(g)$-invariant by the definition.

Observe that the fixed point set for g-action coincides with that for g^{-1}-action. Set

$$H^*_{deloc,G}(B,\mathbb{R}) := \{c = \oplus_{(g),g\in G}\{c_g\} \in H^*_{deloc,G}(B,\mathbb{C}) : \forall g\in G, c_{g^{-1}} = \overline{c_g}\}. \quad (4.8)$$

Let $\Omega^*_{deloc,G}(B,\mathbb{R}) \subset \Omega^*_{deloc,G}(B,\mathbb{C})$ be the ring of forms $\omega = \oplus_{(g),g\in G}\{\omega_g\}$, such that $\forall g \in G, \omega_{g^{-1}} = \overline{\omega_g}$. Then $H^*_{deloc,G}(B,\mathbb{R})$ is the cohomology of the differential complex $(\Omega^*_{deloc,G}(B,\mathbb{R}), d)$. Since $\mathrm{ch}(\phi_{g^{-1}}(\mathcal{K})) = \overline{\mathrm{ch}(\phi_g(\mathcal{K}))}$, from (4.7), for any $\mathcal{K} \in K^*_G(B)$, $\mathrm{ch}_G(\mathcal{K}) \in H^*_{deloc,G}(B,\mathbb{R})$. Thus $\mathrm{ch}_G(K^*_G(B)\otimes\mathbb{R}) \subseteq H^*_{deloc,G}(B,\mathbb{R})$. Since (4.7) is an isomorphism, we obtain a group isomorphism

$$\mathrm{ch}_G : K^*_G(B)\otimes\mathbb{R} \xrightarrow{\simeq} H^*_{deloc,G}(B,\mathbb{R}). \quad (4.9)$$

Definition 4.1. (Compare with [16, Definition 2.4]) A cycle for an equivariant differential K-theory class over B is a pair $(\mathcal{F},\rho)$, where $\mathcal{F} \in \mathrm{F}^*_G(B)$ and $\rho \in \Omega^*_{deloc,G}(B,\mathbb{R})/\mathrm{Im}\, d$. The cycle $(\mathcal{F},\rho)$ is called even (resp. odd) if $\mathcal{F}$ is even (resp. odd) and $\rho \in \Omega^{\mathrm{odd}}_{deloc,G}(B,\mathbb{R})/\mathrm{Im}\, d$ (resp. $\rho \in \Omega^{\mathrm{even}}_{deloc,G}(B,\mathbb{R})/\mathrm{Im}\, d$). Two cycles $(\mathcal{F},\rho)$ and $(\mathcal{F}',\rho')$ are called isomorphic if $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic and $\rho = \rho'$. Let $\widehat{\mathrm{IC}}^0_G(B)$ (resp. $\widehat{\mathrm{IC}}^1_G(B)$) denote the set of isomorphic classes of even (resp. odd) cycles over B with a natural abelian semi-group structure by $(\mathcal{F},\rho)+(\mathcal{F}',\rho') = (\mathcal{F}+\mathcal{F}',\rho+\rho')$.

For $\mathcal{F} \in \mathrm{F}^*_G(B)$, we assume that there exists a perturbation operator A with respect to $D(\mathcal{F})$. For any $g \in G$, by Definition 3.12, the equivariant eta form restricted on the fixed point set of g is $\mathrm{C}_G(g)$-invariant, that is, $\widetilde{\eta}_g(\mathcal{F}|_{B^g}, A|_{B^g}) \in [\Omega^*(B^g,\mathbb{C})]^{\mathrm{C}_G(g)}$. Let h^* be the map in (4.5). Since the perturbation operator A is equivariant, from Definition 3.12, we have

$$\widetilde{\eta}_{g'}(\mathcal{F}|_{B^{g'}}, A|_{B^{g'}}) = h^*\widetilde{\eta}_g(\mathcal{F}|_{B^g}, A|_{B^g}). \quad (4.10)$$

From Definition 3.12, $\widetilde{\eta}_{g^{-1}}(\mathcal{F}|_{B^g}, A|_{B^g}) = \overline{\widetilde{\eta}_g(\mathcal{F}|_{B^g}, A|_{B^g})}$. So the following definition is well-defined.

Definition 4.2. The delocalized eta form $\widetilde{\eta}_G(\mathcal{F},A)$ is defined by

$$\widetilde{\eta}_G(\mathcal{F},A) = \bigoplus_{(g),g\in G}\{\widetilde{\eta}_g(\mathcal{F}|_{B^g}, A|_{B^g})\} \in \Omega^*_{deloc,G}(B,\mathbb{R})/\mathrm{Im}\, d. \quad (4.11)$$

By the same process, we can define

$$\mathrm{FLI}_G(\mathcal{F}) = \bigoplus_{(g),g\in G}\{\mathrm{FLI}_g(\mathcal{F})\} \in \Omega^*_{deloc,G}(B,\mathbb{R}). \quad (4.12)$$

From (3.66), we have

$$d\widetilde{\eta}_G(\mathcal{F},A) = \mathrm{FLI}_G(\mathcal{F}). \quad (4.13)$$

Let $\mathcal{F} \in \mathrm{F}^*_G(B)$ and A be a perturbation operator with respect to $D(\mathcal{F})$. Then by Definition 2.3, there exists a perturbation operator A^{op} with respect to $D(\mathcal{F}^{\mathrm{op}})$ such that

$$\widetilde{\eta}_G(\mathcal{F}^{\mathrm{op}}, A^{\mathrm{op}}) = -\widetilde{\eta}_G(\mathcal{F},A). \quad (4.14)$$

Let $\mathcal{F},\mathcal{F}' \in \mathrm{F}^*_G(B)$, A, A' be perturbation operators with respect to $D(\mathcal{F})$, $D(\mathcal{F}')$ respectively. By Definition 3.12, we have

$$\widetilde{\eta}_G(\mathcal{F}+\mathcal{F}', A\sqcup_B A') = \widetilde{\eta}_G(\mathcal{F},A) + \widetilde{\eta}_G(\mathcal{F}',A'). \quad (4.15)$$

From Remark 3.4, we know that for any $\mathcal{F} \in \mathrm{F}_G^*(B)$, there exists a perturbation operator A with respect to $D(\mathcal{F}+\mathcal{F}^{\mathrm{op}})$ and $A = A^{\mathrm{op}}$. From (4.14), we have

$$\widetilde{\eta}_G(\mathcal{F}+\mathcal{F}^{\mathrm{op}}, A) = 0. \tag{4.16}$$

Definition 4.3. (Compare with [16, Definition 2.10]) We call two cycles $(\mathcal{F}, \rho)$ and $(\mathcal{F}', \rho')$ **paired** if $\mathrm{Ind}(D(\mathcal{F})) = \mathrm{Ind}(D(\mathcal{F}'))$, and there exists a perturbation operator A with respect to $D(\mathcal{F}+\mathcal{F}^{'\mathrm{op}})$ such that

$$\rho - \rho' = \widetilde{\eta}_G(\mathcal{F}+\mathcal{F}^{'\mathrm{op}}, A). \tag{4.17}$$

From (4.14)-(4.16), we have

Lemma 4.4. *(Compare with [16, Lemmas 2.11, 2.12]) The relation "paired" is symmetric, reflexive and compatible with the semigroup structure on* $\widehat{\mathrm{IC}}_G^*(B)$.

Definition 4.5. (Compare with [16, Definition 2.14]) Let $\sim$ denote the equivalence relation generated by the relation "paired". The equivariant differential K-group $\widehat{K}_G^0(B)$ (resp. $\widehat{K}_G^1(B)$) is the group completion of the abelian semigroup $\widehat{\mathrm{IC}}_G^{\mathrm{even}}(B)/\sim$ (resp. $\widehat{\mathrm{IC}}_G^{\mathrm{odd}}(B)/\sim$).

If $(\mathcal{F}, \rho) \in \widehat{\mathrm{IC}}_G^*(B)$, we denote by $[\mathcal{F}, \rho] \in \widehat{K}_G^*(B)$ the corresponding class in equivariant differential K-group. From (4.14)-(4.16), for any $[\mathcal{F}, \rho], [\mathcal{F}', \rho'] \in \widehat{K}_G^*(B)$, we have

$$[\mathcal{F}, \rho] = [\mathcal{F}+\mathcal{F}^{'\mathrm{op}}, \rho-\rho'] + [\mathcal{F}', \rho']. \tag{4.18}$$

So every element of $\widehat{K}_G^*(B)$ can be represented in the form $[\mathcal{F}, \rho]$. Furthermore, we have $-[\mathcal{F}, \rho] = [\mathcal{F}^{\mathrm{op}}, -\rho]$.

4.2 Push-forward map

In this subsection, we construct a well-defined push-forward map in equivariant differential K-theory and prove the functoriality of it using the theorems in Section 3. This solves a question proposed in [16] when $G = \{e\}$. We use the notation in Section 1.4.

Let $\pi_Y : V \to B$ be an equivariant smooth surjective proper submersion of compact G-manifolds with compact orientable fibers Y. We assume that the G-action on B has finite stabilizers only. Thus, so is the action on V. We assume that TY is oriented and π_Y has an equivariant K-orientation in Definition 2.8.

For $g \in G$, the fixed point set V^g is the total space of the fiber bundle $\pi_Y|_{V^g} : V^g \to B^g$ with fibers Y^g. Since the pullback isomorphism h^* in (4.5) commutes with the integral along the fiber, for $\alpha = \oplus_{(g), g\in G}\{\alpha_g\} \in \Omega^*_{deloc,G}(V, \mathbb{R})$, the integral

$$\int_{Y,G} \alpha := \bigoplus_{(g), g\in G} \left\{ \int_{Y^g} \alpha_g \right\} \in \Omega^*_{deloc,G}(B, \mathbb{R}) \tag{4.19}$$

does not depend on $g \in (g)$. So it defines an integral map

$$\int_{Y,G} : \Omega^*_{deloc,G}(V, \mathbb{R}) \to \Omega^*_{deloc,G}(B, \mathbb{R}). \tag{4.20}$$

Consider the set $\widehat{\mathcal{O}}_G^*(\pi_Y)$ of equivariant geometric data $\widehat{o}_Y = (T^H_{\pi_Y}V, g^{TY}, \nabla^{L_Y}, \sigma_Y)$, where $\sigma_Y \in \Omega^{odd}_{deloc,G}(V)/\mathrm{Im}d$.

Let

$$\mathrm{Td}_G(\nabla^{TY}, \nabla^{L_Y}) := \bigoplus_{(g), g\in G} \left\{ \mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y}) \right\} \in \Omega^*_{deloc,G}(V, \mathbb{R}). \tag{4.21}$$

Let $\widehat{o}'_Y=(T^{'H}_{\pi_Y}V,g^{'TY},\nabla^{'L_Y},\sigma'_Y)\in\widehat{\mathcal{O}}^*_G(\pi_Y)$ be another equivariant tuple with the same equivariant K-orientation in Definition 2.8. As in (3.93), from [42, Theorem B.5.4], we can construct the Chern-Simons form $\widetilde{\mathrm{Td}}_G(\nabla^{TY},\nabla^{L_Y},\ \nabla^{'TY},\nabla^{'L_Y})\in\Omega^{odd}_{deloc,G}(V)/\mathrm{Im}d$ such that

$$d\,\widetilde{\mathrm{Td}}_G(\nabla^{TY},\nabla^{L_Y},\nabla^{'TY},\nabla^{'L_Y})=\mathrm{Td}_G(\nabla^{'TY},\nabla^{'L_Y})-\mathrm{Td}_G(\nabla^{TY},\nabla^{L_Y}). \tag{4.22}$$

We introduce a relation $\widehat{o}_Y\sim\widehat{o}'_Y$ as in [16]: two equivariant tuples $\widehat{o}_Y$, $\widehat{o}'_Y$ are related if and only if

$$\sigma'_Y-\sigma_Y=\widetilde{\mathrm{Td}}_G(\nabla^{TY},\nabla^{L_Y},\nabla^{'TY},\nabla^{'L_Y}), \tag{4.23}$$

where we mark the objects associated with the second tuple by $'$.

Definition 4.6. (Compare with [16, Definition 3.5]) The set of equivariant differential K-orientations is the set of equivalence classes $\widehat{\mathcal{O}}^*_G(\pi_Y)/\sim$.

We now start with the construction of the push-forward map $\widehat{\pi}_Y!:\widehat{K}^*_G(V)\to\widehat{K}^*_G(B)$ for a given equivariant differential K-orientation which extends Theorem 2.9 to the differential case. For $[\mathcal{F}_X,\rho]\in\widehat{K}^*_G(V)$, let $\mathcal{F}_Z$ be the equivariant geometric family defined in (2.37). We define (cf. [16, (17)])

$$\begin{aligned}\widehat{\pi}_Y!([\mathcal{F}_X,\rho])=\Big[\mathcal{F}_Z,\int_{Y,G}\mathrm{Td}_G(\nabla^{TY},\nabla^{L_Y})\wedge\rho+\widetilde{\mathrm{FLI}}_G\left(\nabla^{TY,TX},\nabla^{L_Z},\nabla^{TZ},\nabla^{L_Z}\right)\\ +\int_{Y,G}\sigma_Y\wedge(\mathrm{FLI}_G(\mathcal{F}_X)-d\rho)\Big]\in\widehat{K}^*_G(B),\end{aligned} \tag{4.24}$$

where $\widetilde{\mathrm{FLI}}_G:=\bigoplus_{(g),g\in G}\widetilde{\mathrm{FLI}}_g\in\Omega^*_{deloc,G}(B,\mathbb{R})/\mathrm{Im}d$.

Theorem 4.7. *(Compare with [16, Lemma 3.14]) The map $\widehat{\pi}_Y!:\widehat{K}^*_G(V)\to\widehat{K}^*_G(B)$ in (4.24) is well-defined.*

Proof. Let $(\mathcal{F}_X,\rho)$, $(\mathcal{F}'_X,\rho')$ be two cycles over V. By (4.24), we have

$$\widehat{\pi}_Y!(\mathcal{F}_X,\rho)-\widehat{\pi}_Y!(\mathcal{F}'_X,\rho')=\widehat{\pi}_Y!(\mathcal{F}_X+\mathcal{F}^{'\mathrm{op}}_X,\rho-\rho'). \tag{4.25}$$

If $(\mathcal{F}_X,\rho)$ is paired with $(\mathcal{F}'_X,\rho')$, there exists a perturbation operator A, such that

$$\rho-\rho'=\widetilde{\eta}_G(\mathcal{F}_X+\mathcal{F}^{'\mathrm{op}}_X,A). \tag{4.26}$$

So we only need to prove that if there exists a perturbation operator A_X with respect to $D(\mathcal{F}_X)$, $\widehat{\pi}_Y!([\mathcal{F}_X,\widetilde{\eta}_G(\mathcal{F}_X,A_X)])=0\in\widehat{K}^*_G(B)$.

From (4.24), we have

$$\begin{aligned}\widehat{\pi}_Y!([\mathcal{F}_X,\widetilde{\eta}_G(\mathcal{F}_X,A_X)])=\Big[\mathcal{F}_Z,\int_{Y,G}\mathrm{Td}_G(\nabla^{TY},\nabla^{L_Y})\,\widetilde{\eta}_G(\mathcal{F}_X,A_X)\\ +\widetilde{\mathrm{FLI}}_G\left(\nabla^{TY,TX},\nabla^{L_Z},\nabla^{TZ},\nabla^{L_Z}\right)\\ +\int_{Y,G}\sigma_Y\wedge(\mathrm{FLI}_G(\mathcal{F}_X)-d\widetilde{\eta}_G(\mathcal{F}_X,A_X))\Big].\end{aligned} \tag{4.27}$$

From Proposition 3.3 (iii) and Lemma 3.6, there exists a perturbation operator A_Z with respect to $D(\mathcal{F}_Z)$. By Theorem 3.18, (4.13), (4.15) and (4.19), there exists $x\in K^*_G(B)$ such that

$$\widehat{\pi}_Y!(\mathcal{F}_X,\widetilde{\eta}_G(\mathcal{F}_X,A_X))=\left[\mathcal{F}_Z,\widetilde{\eta}_G(\mathcal{F}_Z,A_Z)-\mathrm{ch}_G(x)\right]. \tag{4.28}$$

From Proposition 3.9, if $x \in K^1_G(B)$,there exist $\mathcal{F} \in \mathrm{F}^1_G(B)$ and equivariant spectral sections P, Q with respect to $D(\mathcal{F})$, such that $[P-Q] = x$. Let A_P, A_Q be perturbation operators associated with P, Q respectively. From Theorem 3.17, we have

$$\mathrm{ch}_G(x) = \widetilde{\eta}_G(\mathcal{F}, A_P) - \widetilde{\eta}_G(\mathcal{F}, A_Q). \tag{4.29}$$

If $x \in K^0_G(B)$, by Proposition 3.9, there exist $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{F}^0_G(B)$ and equivariant spectral sections P_1, Q_1 of $D(\mathcal{F}_1)$ and P_2, Q_2 of $D(\mathcal{F}_2)$, such that $x = [P_1 - Q_1] - [P_2 - Q_2]$. Let A_{P_i}, A_{Q_i} be perturbation operators associated with P_i, Q_i for $i = 0, 1$. From Theorem 3.17, letting $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, $A_P = A_{P_1} \sqcup_B A_{Q_2}$, $A_Q = A_{P_2} \sqcup_B A_{Q_1}$, we also have

$$\begin{aligned}\mathrm{ch}_G(x) = \widetilde{\eta}_G(\mathcal{F}_1, A_{P_1}) - \widetilde{\eta}_G(\mathcal{F}_1, A_{Q_1}) - (\widetilde{\eta}_G(\mathcal{F}_2, A_{P_2}) - \widetilde{\eta}_G(\mathcal{F}_2, A_{Q_2}))\\ = \widetilde{\eta}_G(\mathcal{F}_1 + \mathcal{F}_2, A_{P_1} \sqcup_B A_{Q_2}) - \widetilde{\eta}_G(\mathcal{F}_1 + \mathcal{F}_2, A_{P_2} \sqcup_B A_{Q_1})\\ = \widetilde{\eta}_G(\mathcal{F}, A_P) - \widetilde{\eta}_G(\mathcal{F}, A_Q).\end{aligned} \tag{4.30}$$

By (4.14), (4.28)-(4.30) and Definition 4.3, we have

$$\begin{aligned}\widehat{\pi}_Y!(\mathcal{F}_X, \widetilde{\eta}_G(\mathcal{F}_X, A_X)) = \Big[\mathcal{F}_Z, \widetilde{\eta}_G(\mathcal{F}_Z, A_Z) - \widetilde{\eta}_G(\mathcal{F}, A_P) - \widetilde{\eta}_G(\mathcal{F}^{\mathrm{op}}, A_Q^{\mathrm{op}})\Big]\\ = [\mathcal{F} + \mathcal{F}^{\mathrm{op}}, 0] = [\mathcal{F}, 0] - [\mathcal{F}, 0] = 0 \in \widehat{K}^*_G(B).\end{aligned} \tag{4.31}$$

Then from Theorem 2.9, we complete the proof of Theorem 4.7. □

Here our construction of $\widehat{\pi}_Y!$ involve an explicit choice of a representative $\widehat{o}_Y = (T^H_{\pi_Y} V, g^{TY}, \nabla^{L_Y}, \sigma_Y)$ of the equivariant differential K-orientation. In fact, it does not depend on the choice.

Lemma 4.8. *(Compare with [16, Lemma 3.17]) The homomorphism $\widehat{\pi}_Y! : \widehat{K}^*_G(V) \to \widehat{K}^*_G(B)$ only depend on the equivariant differential K-orientation.*

Proof. Let $\widehat{o}_Y = (T^H_{\pi_Y} V, g^{TY}, \nabla^{L_Y}, \sigma_Y)$, $\widehat{o}'_Y = (T'^H_{\pi_Y} V, g'^{TY}, \nabla'^{L_Y}, \sigma'_Y)$ be two representatives of an equivariant differential K-orientation. We will mark the objects associated with the second representative by $'$. From (3.94), we could get

$$\begin{aligned}&\widetilde{\mathrm{FLI}}_G(\nabla^{TY,TX}, \nabla^{L_Z}, \nabla'^{TY,TX}, \nabla'^{L_Z})\\ &\qquad = \int_{Y,G} \widetilde{\mathrm{Td}}_G(\nabla^{TY}, \nabla^{L_Y}, \nabla'^{TY}, \nabla'^{L_Y})\, \mathrm{FLI}_G(\mathcal{F}_X).\end{aligned} \tag{4.32}$$

Then from (4.22), (4.23) and (4.32), we have

$$\begin{aligned}&\widehat{\pi}'_Y!([\mathcal{F}_X, \rho]) - \widehat{\pi}_Y!([\mathcal{F}_X, \rho]) = [\mathcal{F}'_Z + \mathcal{F}^{\mathrm{op}}_Z,\\ &\int_{Y,G} \Big(\mathrm{Td}_G(\nabla'^{TY}, \nabla'^{L_Y}) - \mathrm{Td}_G(\nabla^{TY}, \nabla^{L_Y})\Big) \wedge \rho - \widetilde{\mathrm{FLI}}_G\left(\nabla'^{TZ}, \nabla'^{L_Z}, \nabla'^{TY,TX}, \nabla'^{L_Z}\right)\\ &\quad + \widetilde{\mathrm{FLI}}_G\left(\nabla^{TZ}, \nabla^{L_Z}, \nabla^{TY,TX}, \nabla^{L_Z}\right) + \int_{Y,G} (\sigma'_Y - \sigma_Y) \wedge (\mathrm{FLI}_G(\mathcal{F}_X) - d\rho)\Big]\\ &= \Big[\mathcal{F}'_Z + \mathcal{F}^{\mathrm{op}}_Z, \int_{Y,G} d\, \widetilde{\mathrm{Td}}_G(\nabla^{TY}, \nabla^{L_Y}, \nabla'^{TY}, \nabla'^{L_Y}) \wedge \rho\\ &\quad - \int_{Y,G} \widetilde{\mathrm{Td}}_G(\nabla^{TY}, \nabla^{L_Y}, \nabla'^{TY}, \nabla'^{L_Y}) \wedge d\rho\\ &\quad + \int_{Y,G} \widetilde{\mathrm{Td}}_G(\nabla^{TY}, \nabla^{L_Y}, \nabla'^{TY}, \nabla'^{L_Y})\, \mathrm{FLI}_G(\mathcal{F}_X)\\ &\quad + \widetilde{\mathrm{FLI}}_G\left(\nabla^{TZ}, \nabla^{L_Z}, \nabla'^{TZ}, \nabla'^{L_Z}\right) - \widetilde{\mathrm{FLI}}_G\left(\nabla^{TY,TX}, \nabla^{L_Z}, \nabla'^{TY,TX}, \nabla'^{L_Z}\right)\Big]\end{aligned}$$

$$= [\mathcal{F}'_Z + \mathcal{F}_Z^{\rm op}, \widetilde{\rm FLI}_G\,(\mathcal{F}_Z, \mathcal{F}'_Z)]. \quad (4.33)$$

By Proposition 3.3 (iii) and Lemma 3.6, there exists a perturbation operator A with respect to $D(\mathcal{F}'_Z + \mathcal{F}_Z^{\rm op})$. By Theorem 3.17 and (4.16), there exists $x \in K_G^*(B)$ such that

$$\begin{aligned}\widetilde{\widetilde{\rm FLI}}_G\,(\mathcal{F}_Z, \mathcal{F}'_Z) &= \widetilde{\widetilde{\rm FLI}}_G\,(\mathcal{F}_Z + \mathcal{F}_Z^{\rm op}, \mathcal{F}'_Z + \mathcal{F}_Z^{\rm op}) \\ &= -\widetilde{\eta}_G(\mathcal{F}'_Z + \mathcal{F}_Z^{\rm op}, A) + {\rm ch}_G(x). \quad (4.34)\end{aligned}$$

Following the same process in (4.28)-(4.31), we have $\widehat{\pi}'_Y!([\mathcal{F}_X, \rho]) = \widehat{\pi}_Y!([\mathcal{F}_X, \rho])$.

The proof of Lemma 4.8 is completed. □

We now discuss the functoriality of the push-forward maps with respect to the composition of fiber bundles. Let $\pi_Y : V \to B$ with fibers Y be as in the above subsection together with a representative of an equivariant differential K-orientation $\widehat{o}_Y = (T^H_{\pi_Y} V, g^{TY}, \nabla^{L_Y}, \sigma_Y)$. Let $\pi_U : B \to S$ be another equivariant smooth surjective proper submersion with compact oriented fibers U together with a representative of an equivariant differential K-orientation $\widehat{o}_U = (T^H_{\pi_U} B, g^{TU}, \nabla^{L_U}, \sigma_U)$.

Let $\pi_A := \pi_U \circ \pi_Y : V \to S$ be the composition of two submersions with fibers A. Let $T^H_{\pi_A} V$ be a horizontal subbundle associated with π_A. We assume that $T^H_{\pi_A} V \subset T^H_{\pi_Y} V$. Set $g^{TA} = \pi_Y^* g^{TU} \oplus g^{TY}$, $\nabla^{L_A} = \pi_Y^* \nabla^{L_U} \otimes \nabla^{L_Y}$.

Definition 4.9. (Compare with [16, Definition 3.21]) We define $\widehat{o}_A = \widehat{o}_U \circ \widehat{o}_Y$ by

$$\widehat{o}_A := (T^H_{\pi_A} V, g^{TA}, \nabla^{L_A}, \sigma_A), \quad (4.35)$$

where

$$\begin{aligned}\sigma_A := \sigma_Y \wedge \pi_Y^* \,{\rm Td}_G(\nabla^{TU}, \nabla^{L_U}) + {\rm Td}_G(\nabla^{TY}, \nabla^{L_Y}) \wedge \pi_Y^* \sigma_U \\ + \widetilde{\rm Td}_G(\nabla^{TA}, \nabla^{L_A}, \nabla^{TU,TY}, \nabla^{L_A}) - d\sigma_Y \wedge \pi_Y^* \sigma_U. \quad (4.36)\end{aligned}$$

Theorem 4.10. *(Compare with [16, Theorem 3.23]) We have the equality of homomorphisms* $\widehat{K}_G^*(V) \to \widehat{K}_G^*(S)$

$$\widehat{\pi}_A! = \widehat{\pi}_U! \circ \widehat{\pi}_Y!. \quad (4.37)$$

Proof. The topological part of Theorem 4.10 is just Theorem 2.10 and the differential part follows from a direct calculation using (4.24) and (4.36). □

4.3 Cup product

In this subsection, we construct the cup product in equivariant differential K-theory in our model as in [16, 18] and prove the desired properties.

Let $f : B_1 \to B_2$ be a G-equivariant smooth map. We define the induced homomorphism $f^* : \widehat{K}_G^*(B_2) \to \widehat{K}_G^*(B_1)$ as follows. For $\mathcal{F} \in {\rm F}_G^*(B_2)$, in order to define $f^*\mathcal{F}$, as remarked in Section 2.2, we need take care with the the pullback of the horizontal subbundle. Let F be the natural map from f^*W to W. We choose the new horizontal subbundle $T^H_{f^*\pi}(f^*W)$ by the condition that $dF(T^H_{f^*\pi}(f^*W)) \subseteq T^H_\pi(W)$. Note that the chosen of the new horizontal subbundle is not unique. If A is a perturbation operator with respect to $D(\mathcal{F})$, then f^*A is a perturbation operator with respect to $D(f^*\mathcal{F})$. Moreover, from Definition 3.12, we have

$$\widetilde{\eta}_G(f^*\mathcal{F}, f^*A) = f^*\widetilde{\eta}_G(\mathcal{F}, A). \quad (4.38)$$

By Proposition 3.15 and Definition 4.3, if $\mathcal{F}_1$, $\mathcal{F}_2 \in \mathrm{F}_G^*(B)$ be the pullbacks of $\mathcal{F}$ associated with distinct horizontal subbundles, $(\mathcal{F}_1, 0) \sim (\mathcal{F}_2, 0)$. So we obtain a well defined pullback map

$$f^* : \widehat{K}_G^*(B_2) \to \widehat{K}_G^*(B_1). \tag{4.39}$$

Evidently, $\mathrm{Id}_B^* = \mathrm{Id}_{\widehat{K}_G(B)}$. Let $f' : B_0 \to B_1$ be another equivariant smooth map. We could get

$$f^{'*} f^* = (f \circ f')^* : \widehat{K}_G^*(B_2) \to \widehat{K}_G^*(B_0). \tag{4.40}$$

Let $[\mathcal{F}, \rho] \in \widehat{K}_G^i(B)$ and $[\mathcal{F}', \rho'] \in \widehat{K}_G^*(B)$, where $i = 0, 1$. We define (compare with [16, Definition 4.1])

$$\begin{aligned}[][\mathcal{F}, \rho] \cup [\mathcal{F}', \rho'] := [\mathcal{F} \times_B \mathcal{F}', (-1)^i \mathrm{FLI}_G(\mathcal{F}) \wedge \rho' \\ + \rho \wedge \mathrm{FLI}_G(\mathcal{F}') - (-1)^i d\rho \wedge \rho'].\end{aligned} \tag{4.41}$$

It is obvious that the product is natural with respect to pullbacks.

Proposition 4.11. *(Compare with [16, Propositions 4.2, 4.5]) (i) The product is well defined. It turns $B \mapsto \widehat{K}_G^*(B)$ into a contravariant functor from compact smooth G-manifolds with finite stablizers to unital graded commutative rings. The unit is simply given by $[\mathcal{F}, 0]$, where $\mathcal{F}$ is the equivariant geometric family in Example 2.5 a) such that E_+ is 1 dimensional trivial representation and $E_- = 0$.*

(ii) The product is associative.

(iii) Let $\pi_U : B \to S$ be an equivariant smooth proper submersion with oriented fibers and an equivariant differential K-orientation. For $x \in \widehat{K}_G^*(B)$ and $y \in \widehat{K}_G^*(S)$, we have

$$\widehat{\pi}_U!(\pi_U^* y \cup x) = y \cup \widehat{\pi}_U!(x). \tag{4.42}$$

Proof. The product is obviously biadditive.

From Theorem 3.19 and a direct calculation, we could get the product is compatible with the equivalence relation in differential K-theory. Other properties are the direct extension of the discussions in [16, p47-50]. □

Theorem 4.12. *(Compare with [16, Section 3,4]) The equivariant differential K-theory $\widehat{K}_G$ is a contravariant functor $B \to \widehat{K}_G(B)$ from the category of compact smooth G-manifolds with finite stabilizers to unital $\mathbb{Z}_2$-graded commutative rings together with well-defined transformations*

(1) $R : \widehat{K}_G^*(B) \to \Omega^*_{deloc,G,cl}(B, \mathbb{R})$ (curvature);

(2) $I : \widehat{K}_G^*(B) \to K_G^*(B)$ (underlying K_G-group);

(3) $a : \Omega^*_{deloc,G}(B, \mathbb{R})/\mathrm{Im}\, d \to \widehat{K}_G(X)$ (action of forms),

where $\Omega^*_{deloc,G,cl}(B, \mathbb{R})$ denotes the set of closed delocalized differential forms, such that

(1) the following diagram commutes

$$\begin{array}{ccc} \widehat{K}_G^*(B) & \xrightarrow{\quad I \quad} & K_G^*(B) \\ \Big\downarrow R & & \Big\downarrow \mathrm{ch}_G \\ \Omega^*_{deloc,G,cl}(B, \mathbb{R}) & \xrightarrow{\text{de Rham}} & H^*_{deloc,G}(B, \mathbb{R}); \end{array}$$

(2)

$$R \circ a = d; \tag{4.43}$$

(3) a is of degree 1;

(4) for $x, y \in \widehat{K}_G^*(B)$ and $\alpha \in \Omega^*_{deloc,G}(B,\mathbb{R})/\mathrm{Im} d$, we have

$$R(x \cup y) = R(x) \wedge R(y), \ I(x \cup y) = I(x) \cup I(y), \ a(\alpha) \cup x = a(\alpha \wedge R(x)); \quad (4.44)$$

(5) the following sequence is exact:

$$K_G^{*-1}(B) \xrightarrow{\mathrm{ch}_G} \Omega^{*-1}_{deloc,G}(B,\mathbb{R})/\mathrm{Im}\, d \xrightarrow{a} \widehat{K}_G^*(B) \xrightarrow{I} K_G^*(B) \longrightarrow 0. \quad (4.45)$$

Proof. We define the natural transformation

$$I : \widehat{K}_G^*(B) \to K_G^*(B) \quad (4.46)$$

by

$$I([\mathcal{F}, \rho]) := \mathrm{Ind}(D(\mathcal{F})). \quad (4.47)$$

From Definition 4.3, the transformation I is well defined.

Let a be a parity-reversing natural transformation

$$a : \Omega^{\mathrm{even/odd}}_{deloc,G}(B,\mathbb{R})/\mathrm{Im}\, d \to \widehat{K}_G^{1/0}(B) \quad (4.48)$$

by

$$a(\rho) := [\emptyset, -\rho], \quad (4.49)$$

where $\emptyset$ is the empty geometric family.

We define a transformation

$$R : \widehat{\mathrm{IC}}_G^*(B) \to \Omega^*_{deloc,G,cl}(B,\mathbb{R}) \quad (4.50)$$

by

$$R((\mathcal{F}, \rho)) := \mathrm{FLI}_G(\mathcal{F}) - d\rho. \quad (4.51)$$

If $(\mathcal{F}', \rho')$ is paired with $(\mathcal{F}, \rho)$, there exists a perturbation operator A with respect to $D(\mathcal{F} + \mathcal{F}'^{\mathrm{op}})$, such that $\rho - \rho' = \widetilde{\eta}_G(\mathcal{F} + \mathcal{F}'^{\mathrm{op}}, A)$. From (3.66) and (4.17), we have

$$\begin{aligned} R((\mathcal{F}, \rho)) &= \mathrm{FLI}_G(\mathcal{F}) - d\rho = \mathrm{FLI}_G(\mathcal{F}) - d\rho' - d\widetilde{\eta}_G(\mathcal{F} + \mathcal{F}'^{\mathrm{op}}, A) \\ &= \mathrm{FLI}_G(\mathcal{F}) - d\rho' - \mathrm{FLI}_G(\mathcal{F}) + \mathrm{FLI}_G(\mathcal{F}') = R((\mathcal{F}', \rho')). \end{aligned} \quad (4.52)$$

Since R is additive, it descends to $\widehat{\mathrm{IC}}_G^*(B)/\sim$ and finally to the map $R : \widehat{K}_G^*(B) \to \Omega^*_{deloc,G,cl}(B,\mathbb{R})$. Let $f : B_1 \to B_2$ be a G-equivariant smooth map. It follows from $\mathrm{FLI}_G(f^*\mathcal{F}) = f^*\mathrm{FLI}_G(\mathcal{F})$ that R is natural.

From (4.49) and (4.51), we have

$$R \circ a = d. \quad (4.53)$$

By (3.51), the diagram commutes.

The formulas in (4.44) follow from straight calculations using the definitions.

At last, we prove the exactness of the sequence (4.45).

The surjectivity of I follows from Proposition 2.6.

Next, we show the exactness at $\widehat{K}_G^*(B)$. It is obvious that $I \circ a = 0$. For a cycle $(\mathcal{F}, \rho)$, if $I([\mathcal{F}, \rho]) = 0$, we have $\mathrm{Ind}(D(\mathcal{F})) = 0$. By Example 2.5 b), we could take $\mathcal{F}$ such that at least one component of the fiber has the nonzero dimension. So there exists a perturbation operator A with respect to $D(\mathcal{F})$ from Proposition 3.3. By (4.17) and (4.49), we have

$$[\mathcal{F}, \rho] = a(\widetilde{\eta}_G(\mathcal{F}, A) - \rho). \tag{4.54}$$

Finally, we prove the exactness at $\Omega^{*-1}_{deloc,G,cl}(B,\mathbb{R})/\operatorname{Im} d$. Following the same process in (4.28)-(4.31), for any $x \in K^*_G(B)$, by (4.49),

$$a \circ \mathrm{ch}_G(x) = (\emptyset, \tilde{\eta}_G(\mathcal{F}, A_Q) - \tilde{\eta}_G(\mathcal{F}, A_P)) = [\mathcal{F}, 0] - [\mathcal{F}, 0] = 0. \tag{4.55}$$

If $a(\rho) = 0$, for any equivariant geometric family $\mathcal{F}$ with a perturbation operator A with respect to $D(\mathcal{F})$, by Definition 4.3 and (4.54), we have

$$[\mathcal{F}, \tilde{\eta}_G(\mathcal{F}, A) - \rho] = a(\rho) = 0 = [\mathcal{F}, \tilde{\eta}_G(\mathcal{F}, A)]. \tag{4.56}$$

So by Definition 4.5, there exists another cycle $(\mathcal{F}', \rho')$, such that $(\mathcal{F} + \mathcal{F}', \rho' + \tilde{\eta}_G(\mathcal{F}, A) - \rho) \sim (\mathcal{F} + \mathcal{F}', \rho' + \tilde{\eta}_G(\mathcal{F}, A))$. Since $\sim$ is generated by "paired", we have the cycles $\{(\mathcal{F}_i, \rho_i)\}_{0 \leqslant i \leqslant r}$ such that for any $1 \leqslant i \leqslant r$, $(\mathcal{F}_i, \rho_i)$ is paired with $(\mathcal{F}_{i-1}, \rho_{i-1})$, $(\mathcal{F}_0, \rho_0) = (\mathcal{F} + \mathcal{F}', \rho' + \tilde{\eta}_G(\mathcal{F}, A) - \rho)$ and $(\mathcal{F}_r, \rho_r) = (\mathcal{F} + \mathcal{F}', \rho' + \tilde{\eta}_G(\mathcal{F}, A))$. By Definition 4.3, for any $1 \leqslant i \leqslant r$, there exists a perturbation operator A_i with respect to $D(\mathcal{F}_{i-1} + \mathcal{F}_i^{\mathrm{op}})$ such that $\rho_{i-1} - \rho_i = \tilde{\eta}_G(\mathcal{F}_{i-1} + \mathcal{F}_i^{\mathrm{op}}, A_i)$. Let A_i' $(0 \leqslant i \leqslant r)$ be the perturbation operator with respect to $D(\mathcal{F}_i + \mathcal{F}_i^{\mathrm{op}})$ taken in (4.16). Therefore, by Theorem 3.17, (4.15) and (4.16), there exists $x \in K^*_G(B)$, such that

$$\begin{aligned} -\rho = \sum_{i=1}^{r} (\rho_{i-1} - \rho_i) &= \tilde{\eta}_G(\mathcal{F}_0 + \mathcal{F}_1^{\mathrm{op}} + \cdots + \mathcal{F}_{r-1} + \mathcal{F}_r^{\mathrm{op}}, A_1 \sqcup_B \cdots \sqcup_B A_r) \\ &= \tilde{\eta}_G(\mathcal{F}_0 + \mathcal{F}_r^{\mathrm{op}} + \cdots + \mathcal{F}_{r-1} + \mathcal{F}_{r-1}^{\mathrm{op}}, A_1 \sqcup_B \cdots \sqcup_B A_r) \\ &\quad - \tilde{\eta}_G(\mathcal{F}_0 + \mathcal{F}_r^{\mathrm{op}} + \cdots + \mathcal{F}_{r-1} + \mathcal{F}_{r-1}^{\mathrm{op}}, A_0' \sqcup_B \cdots \sqcup_B A_{r-1}') = \mathrm{ch}_G(x). \end{aligned} \tag{4.57}$$

The proof of Theorem 4.12 is completed. □

The direct extension of [16, Proposition 3.19 and Lemma 3.20] show that the pullback map and the exact sequence (4.45) are compatible with the push-forward maps.

Remark 4.13. If the group G is trivial, all the models of differential K-theory are isomorphic (see e.g. [17]). For equivariant case, the uniqueness is an open question.

4.4 Differential K-theory for orbifolds

In [18], Bunke and Schick constructed the first model of the differential K-theory for orbifolds by using the language of stacks and proved the desired properties. It could be regarded as a model of the equivariant differential K-theory when the action has finite stabilizers. In the subsections above, inspired by the constructions in [16, 46], we construct the a model of the equivariant differential K-theory when the action has finite stabilizers. In this subsection, we will explain that this model could also be regarded as a model for orbifolds.

Let $\mathcal{X}$ be a compact orbifold (effective orbifold in some literatures). There exist a compact smooth manifold B and a compact Lie group G such that $\mathcal{X}$ is diffeomorphic to a quotient for a smooth effective G-action on B with finite stabilizers (see [1, Theorem 1.23]).

Let $K^0_{orb}(\mathcal{X})$ be the orbifold K^0-group of the compact orbifold $\mathcal{X}$ defined as the Grothendieck ring of the equivalence classes of orbifold vector bundles over $\mathcal{X}$. Since $\mathcal{X}$ is an orbifold, $\mathcal{X} \times S^1$ is an orbifold. Moreover, $i : \mathcal{X} \to \mathcal{X} \times S^1$ is a morphism in the category of orbifolds. As in (2.19), we define the orbifold K^1 group $K^1_{orb}(\mathcal{X}) := \ker(i^* : K^0_{orb}(\mathcal{X} \times S^1) \to K^0_{orb}(\mathcal{X}))$.

Let $p : B \to B/G = \mathcal{X}$ be the projection. Then from [1, Proposition 3.6], it induces an isomorphism $p^* : K^*_{orb}(\mathcal{X}) \to K^*_G(B)$. Note that if the orbifold $\mathcal{X}$ can be presented in two different ways as a quotient, say $B'/G' \simeq \mathcal{X} \simeq B/G$, it shows that $K^*_{G'}(B') \simeq K^*_{orb}(\mathcal{X}) \simeq K^*_G(B)$. So we can consider the orbifold K-theory as a special case of the equivariant K-theory.

Furthermore, from the definition of the differential structure on orbifolds, we know that $\Omega^*_{deloc,G}(B, \mathbb{R})/\mathrm{Im}d \simeq \Omega^*_{deloc,G'}(B', \mathbb{R})/\mathrm{Im}d$. From the exact sequence in (4.45) and five lemma, we have

$$\widehat{K}^*_{G'}(B') \simeq \widehat{K}^*_G(B). \tag{4.58}$$

Therefore, this model of equivariant differential K-theory for G-action with finite stabilizers could be regarded as a model of differential K-theory for orbifolds.

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Appendix A Equivariant K-theory for smooth complex vector bundles

In this appendix, we show that for a compact Lie group G and a smooth compact manifold B with a smooth G-action, $K_G^0(B)$ in [48] defined by G-equivariant topological complex vector bundles could be studied using the G-equivariant smooth complex vector bundles. Although this is certainly well-known (see e.g., [21, (2.1)]), we were unable to find an explicit proof in the literature. We state it here for the completeness following the suggestion of a referee. In this appendix, all vector bundles are complex.

For a representation V of G, for $v \in V$, if Gv generates a finite dimensional subspace of V, we say v is a G-finite vector in V.

Let E be a G-equivariant smooth vector bundle over B. Take a Hermitian metric on E and let $\|\cdot\|_{\mathcal{C}^0(B,E)}$ be the corresponding $\mathcal{C}^0$-norm. For $s \in \mathcal{C}^\infty(X,E)$, $gs \in \mathcal{C}^\infty(X,E)$. Thus for any $f \in \mathcal{C}^\infty(G)$,

$$s_f := \int_G f(g) gs\, dg \in \mathcal{C}^\infty(B,E), \tag{A.1}$$

where dg is the Haar measure. Since G acts continuously on $\mathcal{C}^\infty(B,E)$, for any $\varepsilon > 0$, there exists a neighborhood U of the unity $e \in G$ such that for any $g \in U$, $\|gs - s\|_{\mathcal{C}^0(B,E)} < \varepsilon/2$. Let $v \in \mathcal{C}^\infty(G)$ be a non-negative function vanishing outside U with $\int_G v(g)dg = 1$. Then $\|s_v - s\|_{\mathcal{C}^0(B,E)} < \varepsilon/2$. Let $M_s := \|\int_G gs\, dg\|_{\mathcal{C}^0(B,E)}$. From Peter-Weyl theorem, there exists a G-finite vector $u \in \mathcal{C}^\infty(G)$, such that $\|v - u\|_{\mathcal{C}^0(G)} < \frac{\varepsilon}{2M_s}$. Thus, $\|s_v - s_u\| < \varepsilon/2$. Observe that s_u is a G-finite vector in $\mathcal{C}^\infty(B,E)$. We have the following lemma.

Lemma A.1. *(cf. [45, §2.16]) For $s \in \mathcal{C}^\infty(B,E)$, for any $\varepsilon > 0$, there exists a G-finite vector $s' \in \mathcal{C}^\infty(B,E)$ such that $\|s - s'\|_{\mathcal{C}^0(B,E)} < \varepsilon$.*

For a finite dimensional complex representation M of G, we consider the G-action on $B \times M$ given by

$$g(b,u) = (gb, gu), \quad \forall g \in G, b \in B, u \in M. \tag{A.2}$$

Thus $B \times M \to B$ is an equivariant smooth vector bundle over B. In this case, a G-invariant Hermitian inner product on M forms a G-invariant Hermitian smooth metric on this vector bundle. The following lemma extends [48, Proposition 2.4] to the category of G-equivariant smooth vector bundles.

Proposition A.2. *Let E be a G-equivariant smooth vector bundle over B. There exist a finite dimensional complex representation M of G and a G-equivariant smooth vector bundle F over B such that $E \oplus F$ is isomorphic to $B \times M$ as G-equivariant smooth vector bundles.*

Proof. It suffices to find a equivariant smooth surjection from some $B \times M$ to E. Then F is the orthogonal complement of E in $B \times M$.

For any $b \in B$, we can choose a finite set $\sigma_b \subset \mathcal{C}^\infty(B,E)$, such that $\{s(b)\}_{s\in\sigma_b}$ spans E_b. From Lemma A.1, we can choose σ_b such that it consists of G-finite vectors in $\mathcal{C}^\infty(B,E)$. There exists a neighborhood of b, U_b, such that for any $x \in U_b$, $\{s(x)\}_{s\in\sigma_b}$ spans E_x. Suppose $U_{b_1}, \cdots, U_{b_m}$ covers B. Let $\sigma = \cup_i \sigma_{b_i}$. Let M be the finite dimensional subspace of $\mathcal{C}^\infty(B,E)$ generated by σ. Then the evaluation map $B \times M \to E$ is the required surjection. □

Lemma A.3. *(Compare with [30, Theorem 3.5]) For every G-equivariant topological vector bundle E over B, there exists a G-equivariant smooth vector bundle E^s over B, which is unique up to isomorphism of G-equivariant smooth vector bundles, such that E^s is isomorphic to E as G-equivariant topological vector bundles.*

Proof. By [48, Proposition 2.4], the $\mathcal{C}^0$-version of Proposition A.2, there exists a finite dimensional complex representation M of G and an equivariant embedding $i : E \to B \times M$. Let $Gr_{M,r}$ be the Grassmannian parameterizing all complex linear subspaces of finite dimensional complex representation M of given dimension r. Since G acts linearly on M, there is a induced smooth G-action on smooth manifold $Gr_{M,r}$. Let r be the rank of E. Define continuous map $h : B \to Gr_{M,r}$ by $h(b) := i(E_b) \in Gr_{M,r}$, $\forall b \in B$. Since i is equivariant, h is an equivariant map. Let $\gamma_{M,r}$ be the universal bundle over $Gr_{M,r}$, which is an equivariant smooth vector bundle over $Gr_{M,r}$. Then $h^*\gamma_{M,r}$ is isomorphic to E as G-equivariant topological vector bundles.

For equivariant continuous map $h : B \to Gr_{M,r}$, there exists an equivariant smooth map $h_G : B \to Gr_{M,r}$, which is G-homotopy to h continuously (see e.g., [13, Theorem VI.4.2]). So $E^s := h_G^*\gamma_{M,r}$ is a G-equivariant smooth vector bundle which is isomorphic to $h^*\gamma_{M,r} \simeq E$ as G-equivariant topological vector bundles.

For two G-equivariant smooth vector bundles E_1^s and E_2^s, which are isomorphic as G-equivariant topological vector bundles, there exist G-equivariant smooth maps $h_i : B \to Gr_{M,r}$, $i = 1, 2$, such that $E_i^s = h_i^*\gamma_{M,r}$ and h_1 is G-homotopy to h_2 continuously. Since G is compact, h_1 is G-homotopy to h_2 smoothly (see e.g., [13, Corollary VI.4.3]). Thus E_1^s is isomophic to E_2^s as G-equivariant smooth vector bundles.

The proof of Lemma A.3 is completed. □

Proposition A.4. *Let $K^0_{G,sm}(B)$ be the Grothendieck group of the G-equivariant smooth vector bundles over B. We have*

$$K^0_{G,sm}(B) \simeq K^0_G(B). \tag{A.3}$$

Proof. Forgetting the smooth structure, we obtain a well-defined map $A : K^0_{G,sm}(B) \to K^0_G(B)$. Let $\mathrm{Vect}_G(B)$ and $\mathrm{Vect}_{G,sm}(B)$ be the equivalence classes of G-equivariant topological and smooth vector bundles over B. Then Lemma A.3 induces a well-defined map $\mathrm{Vect}_G(B) \to \mathrm{Vect}_{G,sm}(B)$. For E_1, E_2 in $\mathrm{Vect}_G(B)$, if $[E_1] = [E_2] \in K^0_G(B)$, there exists topological vector bundle F such that $E_1 \oplus F$ is isomorphic to $E_2 \oplus F$ as G-equivariant topological vector bundles. Let E_1^s, E_2^s and F^s be the corresponding G-equivariant smooth vector bundles. Since $E_1^s \oplus F^s$ is isomorphic to $E_2^s \oplus F^s$ as G-equivariant topological vector bundles, from the uniqueness in Lemma A.3, they are isomorphic as G-equivariant smooth vector bundles. Thus we get a well-defined map $B : K^0_G(B) \to K^0_{G,sm}(B)$. Easy to see that $A \circ B$ and $B \circ A$ are all identity maps.

The proof of Proposition A.4 is completed.

□

Appendix B Equivariant family index for odd dimensional fibers

In this appendix, we summarize some results on the equivariant family index for odd dimensional fibers and the explanations for $K_G^1(B)$ (cf. [4, 26, 44]).

We consider the equivariant $\mathbb{Z}_2$-graded Hilbert bundle $\mathcal{E}$ with fiber $L^2(Z_b, E)$ over $b \in B$. From [26, Lemma A.32], there exists an equivariant embedding from $\mathcal{E}$ to the equivariant trivial Hilbert bundle $B \times L^2(G) \otimes C(\mathbb{R}) \otimes H$, where $C(\mathbb{R})$ is the complex Clifford algebra and H is a separable Hilbert space. As in [26, Definitions A.39 and A.40], for any equivariant $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$, we denote by $\mathrm{Fred}^0(\mathcal{H})$ the space of odd skew-adjoint equivariant Fredholm operators A, for which A^2+1 is compact, topologized as a subspace of $B(\mathcal{H}) \times K(\mathcal{H})$, where $B(\mathcal{H})$ and $K(\mathcal{H})$ are the sets of bounded linear operators and compact operators on $\mathcal{H}$ given the compact-open topology and the norm topology respectively. Denote by $\mathrm{Fred}^1(\mathcal{H})$ the subspace of $\mathrm{Fred}^0(C(\mathbb{R}) \otimes \mathcal{H})$ consisting of odd operators A, which supercommute with the action of $C(\mathbb{R})$ and for which the essential spectrum of $-\sqrt{-1}c(e)A$ contains both positive and negative eigenvalues, where $c(e)$ is the basis element of $C(\mathbb{R})$. By [26, §3.5.4], $K_G^1(B)$ is realized as the space of G-homotopy classes of G-equivariant maps from B to $\mathrm{Fred}^1(L^2(G) \otimes H)$:

$$K_G^1(B) \simeq [B, \mathrm{Fred}^1(L^2(G) \otimes H)]_G. \tag{B.1}$$

Let $T = D(\mathcal{F})/(1+D(\mathcal{F})^2)^{1/2}$. Then T is bounded, G-equivariant and $\mathrm{Ind}(T) = \mathrm{Ind}(D(\mathcal{F})) \in K_G^1(B)$. Moreover $\sqrt{-1}T$ can be extended to an equivariant map from B to $\mathrm{Fred}^1(L^2(G) \otimes H)$ by taking the identity map on the complement of $\mathcal{E}$ in $B \times L^2(G) \otimes C(\mathbb{R}) \otimes H$. By [26, §3.5.4 and Proposition A.41], $\mathrm{Ind}(D(\mathcal{F})) = \mathrm{Ind}(T) \in K_G^1(B)$ corresponds to the element of $K_G^0(B \times (0, \frac{1}{2})) \simeq [B \times (0, \frac{1}{2}), \mathrm{Fred}'^0(L^2(G) \otimes H)]_G$ given by

$$D(\theta) = \cos(2\pi\theta) + \sqrt{-1}T\sin(2\pi\theta), \quad \theta \in (0, \frac{1}{2}). \tag{B.2}$$

Here $\mathrm{Fred}'^0(L^2(G) \otimes H)$ consists of the elements in $\mathrm{Fred}^0(L^2(G) \otimes H)$ which is invertible outside a compact set of the parameter space (see also [4, p6], [44, (3.1)]). By applying the natural inclusion $K_G^0(B \times (0, \frac{1}{2})) \to K_G^0(B \times S^1)$, we obtain an element of $K_G^0(B \times S^1)$ which lies in the image of j in (2.19), thus an element of $K_G^1(B)$.

The following proposition is the equivariant version of [44, Proposition 6]. We prove it here using the notation in Example 2.5 d) for the completeness.

Proposition B.1. *For $\mathcal{F} \in \mathrm{F}_G^1(B)$, there exists inclusion $i : B \to B \times S^1$ such that $i^* \mathrm{Ind}(D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L)) = 0$. Moreover, as an element of $K_G^1(B)$, we have*

$$j\big(\mathrm{Ind}(D(\mathcal{F}))\big) = \mathrm{Ind}(D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L)). \tag{B.3}$$

Proof. In order to compare our definition with that in [4], we replace the connection in (2.28) by $\nabla^L = d + 2\pi(\theta - 1/4)\sqrt{-1}dt$. Since from (2.4) and (2.5),

$$D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L) = D(\mathcal{F}) \otimes J + D(\mathcal{F}^L) \otimes K, \tag{B.4}$$

the index in the right hand side of (B.3) does not vary in $K_G^0(B \times S^1)$ after the replacement. Then we could calculate that for $\theta \in [0, 1)$, the eigenvalues of $D(\mathcal{F}^L)$ are $\{\lambda_k = 2\pi k + 2\pi(\theta - \frac{1}{4})\}_{k\in\mathbb{Z}}$ and the eigenspace of λ_k is one dimensional for any $k \in \mathbb{Z}$. Let s be a local frame of L. The eigenfunction of λ_k is $v_k(t) = \exp(2\pi k\sqrt{-1}t)s$. From (B.4), $D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L)^2 = (D(\mathcal{F})^2 + D(\mathcal{F}^L)^2) \otimes \mathrm{Id}_{\mathbb{C}^2}$.

So it is invertible if and only if $\theta \neq 1/4$. Thus for inclusion $i : B \to B \times S^1$, $i(B) = B \times \{\frac{1}{2}\}$, $i^* \operatorname{Ind}(D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L)) = 0$.

Fix $b \in B$. Since $\mathcal{S}_{S^1\times Z} = \mathcal{S}_{S^1} \otimes \mathcal{S}_Z \otimes \mathbb{C}^2$, we have

$$L^2(S_t^1 \times Z_b, \mathcal{S}_{S^1\times Z}\widehat{\otimes}L\widehat{\otimes}E) = {}'\bigoplus_{k\in\mathbb{Z}} \mathbb{C}v_k(t) \otimes L^2(Z_b, \mathcal{S}_Z\widehat{\otimes}E) \otimes \mathbb{C}^2. \tag{B.5}$$

Here ${}'\oplus$ stands for the direct sum in the category of Hilbert spaces. If $k \neq 0$, $D(p_1^*\mathcal{F}\times_{B\times S^1} p_2^*\mathcal{F}^L)^2|_{\mathbb{C}v_k(t)\otimes L^2(Z_b,\mathcal{S}_Z\widehat{\otimes}E)\otimes\mathbb{C}^2} > 0$. Let $H_+ = \mathbb{C}v_0(t)\otimes L^2(Z_b, \mathcal{S}_Z\widehat{\otimes}E)\otimes(\mathbb{C}\oplus\{0\})$, $H_- = \mathbb{C}v_0(t) \otimes L^2(Z_b, \mathcal{S}_Z\widehat{\otimes}E) \otimes (\{0\}\oplus\mathbb{C})$. Then

$$\operatorname{Ind}(D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L)) = \operatorname{Ind}\left(D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L)_+ : H_+ \to H_-\right). \tag{B.6}$$

We define the isomorphisms $\phi_+ : H_+ \to L^2(Z_b, \mathcal{S}_Z\widehat{\otimes}E)$, $\quad \phi_- : H_- \to L^2(Z_b, \mathcal{S}_Z\widehat{\otimes}E)$, by $\phi_+(v_0(t)\otimes l\otimes(1,0)) = l$, $\quad \phi_-(v_0(t)\otimes l\otimes(0,1)) = l$. Set $D_+ := \phi_- \circ D(p_1^*\mathcal{F}\times_{B\times S^1} p_2^*\mathcal{F}^L)_+ \circ \phi_+^{-1}$. Then on $L^2(Z_b, \mathcal{S}_Z\widehat{\otimes}E)$, from (2.4) and (B.4), we have

$$D_+ = \sqrt{-1}D(\mathcal{F}) + \lambda_0(\theta). \tag{B.7}$$

From (B.2) and (B.7), for $\theta \in (0, \frac{1}{2})$, we have

$$\begin{aligned}\operatorname{Ind}(D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L)) &= \operatorname{Ind}\left(\frac{D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L)_+}{\sqrt{1 + D(p_1^*\mathcal{F} \times_{B\times S^1} p_2^*\mathcal{F}^L)^2}} : H_+ \to H_-\right)\\ &= \operatorname{Ind}\left(\frac{D_+}{\sqrt{1+\lambda_0(\theta)^2 + D(\mathcal{F})^2}}\right) = \operatorname{Ind}\left(\frac{\lambda_0(\theta) + \sqrt{-1}D(\mathcal{F})}{\sqrt{1+\lambda_0(\theta)^2 + D(\mathcal{F})^2}}\right)\\ &= \operatorname{Ind}\left(D(\theta)\right).\end{aligned} \tag{B.8}$$

Since $D(p_1^*\mathcal{F}\times_{B\times S^1} p_2^*\mathcal{F}^L)$ and $D(\theta)$ are invertible for $\theta \neq \frac{1}{4}$, we obtain Proposition B.1. □